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Metastability for Reversible Probabilistic Cellular Automata with Self-Interaction

Emilio N.M. Cirillo · Francesca R. Nardi · Cristian Spitoni

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Abstract The problem of metastability for a stochastic dynamics with a parallel updating rule is addressed in the Freidlin–Wentzel regime, namely, finite volume, small magnetic field, and small temperature. The model is characterized by the existence of many fixed points and cyclic pairs of the zero temperature dynamics, in which the system can be trapped in its way to the stable phase. Our strategy is based on recent powerful approaches, not needing a complete description of the fixed points of the dynamics, but relying on few model dependent results. We compute the exit time, in the sense of logarithmic equivalence, and characterize the critical droplet that is necessarily visited by the system during its excursion from the metastable to the stable state. We need to supply two model dependent inputs: (1) the communication energy, that is the minimal energy barrier that the system must overcome to reach the stable state starting from the metastable one; (2) a recurrence property stating that for any configuration different from the metastable state there exists a path, starting from such a configuration and reaching a lower energy state, such that its maximal energy is lower than the communication energy.

Keywords Stochastic dynamics · Probabilistic cellular automata · Metastability · Low temperature dynamics

E.N.M. Cirillo (✉)
Dipartimento Me. Mo. Mat., Università di Roma La Sapienza, via A. Scarpa 16, 00161 Roma, Italy
e-mail: cirillo@dmmm.uniroma1.it

F.R. Nardi
Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: F.R.Nardi@tue.nl

C. Spitoni
Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands
e-mail: spitoni@eurandom.tue.nl

F.R. Nardi · C. Spitoni
Eurandom, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

1 Introduction

Metastable states are very common in nature and are typical of systems close to a first order phase transition. It is often observed that a system can persist for a long period of time in a phase which is not the one favored by the thermodynamic parameters; classical examples are the super-saturated vapor and the magnetic hysteresis. The rigorous description of this phenomenon in the framework of well defined mathematical models is relatively recent, dating back to the pioneering paper [3], and has experienced substantial progress in the last decade. See [12] for a list of the most important papers on this subject.

A natural setup in which the phenomenon of metastability can be studied is that of Markov chains, or Markov processes, describing the time evolution of a statistical mechanical system. Think for instance to a stochastic lattice spin system. In this context powerful theories (see [2, 9, 11]) have been developed with the aim to find answers valid with maximal generality and to reduce to a minimum the number of model dependent inputs necessary to describe the metastable behavior of the system. Whatever approach is chosen, the key model dependent question is the computation of the minimal energy barrier, called *communication energy*, to be overcome by a path connecting the metastable to the stable state. Such a problem is in general quite complicated and becomes particularly difficult when the dynamics has a parallel character. Indeed, if simultaneous updates are allowed on the lattice, then no constraint on the structure of the trajectories in the configuration space is imposed. Therefore, to compute the communication energy, one must take into account all the possible transitions in the configuration space.

The problem of the computation of the communication energy in a parallel dynamics setup has been addressed in [4, 5]. In particular, in [5] the typical questions of metastability, that is the determination of the *exit time* and of the *exit tube*, have been answered for a reversible Probabilistic Cellular Automaton (see [6, 8, 10, 13–15]), in which each spin is coupled only with its nearest neighbors. In that paper it has been shown that, during the transition from the metastable minus state to the stable plus state, the system visits an intermediate chessboard-like phase. In the present paper we study the reversible PCA in which each spin interacts both with itself and with its nearest neighbors; the metastable behavior of such a model has been investigated on heuristic and numerical grounds in [1]. The addition of the self-interaction changes completely the metastability scenario; in particular we show that the chessboard-like phase plays no role in the exit from the metastable phase.

Another very interesting feature of this model is the presence of a large number of fixed points of the zero-temperature dynamics in which the system can be trapped. Following the powerful approach of [9], we can compute the exit time avoiding a complete description of the trapping states. However, we cannot describe the exit tube, i.e., the tube of trajectories followed by the system during its exit from the metastable to the stable phase. The only information on the exit path that we prove in this paper is the existence of a particular set of configurations which is necessarily visited by the system during its excursion from the metastable to the stable state. This set plays the role of the saddle configuration set, which is usually introduced in the study of the metastable behavior of sequential dynamics.

According to the approach of [9], the model dependent ingredients that must be provided are essentially two: (1) the solution of the global variational problem for all the paths connecting the metastable and the stable state, i.e., the computation of the communication energy; (2) a sort of recurrence property stating that, starting from each configuration different from the metastable and the stable state, it is possible to reach a configuration at lower energy following a path with an energy cost strictly smaller than the communication energy.

To solve the global variational problem (see items 2 and 3 in Theorem 2.3), we obtain an upper bound on the communication energy by exhibiting a path connecting the metastable state to the stable state whose maximal energy is equal to the communication energy. To find the lower bound, we perform a partition of the configuration space, study the transitions between configurations in these partitions, and reduce the computation to the optimal one (see Fig. 8). To prove the recurrence property (see item 1 in Theorem 2.3), we have to face the problem of the existence of a large number of fixed points of the dynamics. We solve this problem by showing that, for each configuration different from the metastable state, it is possible to find a path connecting it to the stable state, i.e., to the unique global minimum of the energy, such that the energy along this path is strictly smaller than the communication energy.

We finally give a brief description of the content of the paper. In Sect. 2 we define the model and state our main result in Theorem 2.1. The proof of Theorem 2.1, based on the model dependent results in Theorem 2.3 and on [9], is given in Sect. 2.8. Section 3 is devoted to the proof of the estimates on the energy landscape stated in Theorem 2.3, namely, the global variational problem (items 2 and 3) and the recurrence property (item 1). The proof of items 2 and 3 relies on Proposition 3.2, which is proven in Sect. 4. The Appendix is devoted to a brief review of results in [9].

2 Model and Results

In this section we introduce the basic notation, define the model, and state our main result. In particular, Sects. 2.1–2.4 are devoted to the definition of the Probabilistic Cellular Automaton which will be studied in the sequel. In Sect. 2.5 we state Theorem 2.1 with the results on the metastable behavior of the system. In Sect. 2.6 we introduce the transition rates and the zero temperature dynamics; in Sect. 2.7 we develop an heuristic argument on which the proof of the theorem is based. In Sect. 2.8, finally, we prove Theorem 2.1.

2.1 The Lattice

The spatial structure is modeled by the two-dimensional finite square $\Lambda := \{0, \dots, L-1\}^2$, where L is a positive integer, with periodic boundary conditions; note that Λ is a torus. We shall use the metric induced by the Euclidean distance on the flat torus. An element of Λ is called a *site*. We use $X^{\mathrm{c}} := \Lambda \setminus X$ to denote the complement of $X \subset \Lambda$.

Let $x \in \Lambda$; we say that $y \in \Lambda$ is a *nearest neighbor* of x if and only if the distance on the torus of x from y is equal to 1. For $X \subset \Lambda$, we say that $y \in X^{\mathrm{c}}$ is an element of the *external boundary* ∂X of X if and only if at least one of its nearest neighbors belongs to X; we let also $\overline{X} := X \cup \partial X$ be the *closure* of X. Two sets $X, Y \subset \Lambda$ are said to be *not interacting* if and only if for any $x \in X$ and $y \in Y$ their distance on the torus is larger or equal to $\sqrt{5}$.

Let $x = (x_1, x_2) \in \Lambda$; for ℓ_1, ℓ_2 positive integers we let $Q_{\ell_1,\ell_2}(x)$ be the collection of the sites $((x_1 + s_1) \bmod L, (x_2 + s_2) \bmod L)$ for $s_i = 0, \dots, x_i + \ell_i - 1$ where $i = 1, 2$. Roughly speaking, $Q_{\ell_1,\ell_2}(x)$ is the rectangle on the torus of side lengths ℓ_1 and ℓ_2 drawn starting from x and moving in the positive direction along the two coordinate axes. For ℓ a positive integer we let $Q_\ell(x) := Q_{\ell,\ell}(x)$.

2.2 The Configuration Space

The single spin state space is given by the finite set $\{-1, +1\}$; the configuration space in $X \subset \Lambda$ is defined as $\mathcal{S}_X := \{-1, +1\}^X$ and considered equipped with the discrete topology and the corresponding Borel σ algebra $\mathcal{F}_X$. The model and the related quantities that will be introduced later on will all depend on Λ, but since Λ is fixed it will be dropped from the notation; in this spirit we let $\mathcal{S}_\Lambda =: \mathcal{S}$ and $\mathcal{F}_\Lambda =: \mathcal{F}$.

Given a configuration $\sigma \in \mathcal{S}$ and $X \subset \Lambda$, we denote by σ_X the *restriction* of σ to X. Let m be a positive integer and let $X_1, \ldots, X_m \subset \Lambda$ be pairwise disjoint subsets of Λ; for $\sigma_k \in \mathcal{S}_{X_k}$, with $k = 1, \ldots, m$, we denote by $\sigma_1\sigma_2\cdots\sigma_m$ the configuration in $\mathcal{S}_{X_1\cup\cdots\cup X_m}$ such that $(\sigma_1\sigma_2\cdots\sigma_m)_{X_k} = \sigma_k$ for all $k \in \{1, \ldots, m\}$. Moreover, given $\sigma \in \mathcal{S}$ and $x \in \Lambda$, we denote by σ^x the configuration such that $\sigma^x(x) = -\sigma(x)$ and $\sigma^x(y) = \sigma(y)$ for $y \neq x$. Let $x \in \Lambda$, we define the shift Θ_x acting on $\mathcal{S}$ by setting $(\Theta_x\sigma)_y := \sigma_{y+x}$ for all $y \in \Lambda$ and $\sigma \in \mathcal{S}$.

Given a function $f : \mathcal{S} \to \mathbb{R}$, if $f \in \mathcal{F}_X$ we shall sometimes write $f(\sigma_X)$ for $f(\sigma)$. Let $f, g : \mathcal{S} \to \mathcal{S}$ be two functions, we consider the *product* or *composed* function $fg : \mathcal{S} \to \mathcal{S}$ such that $fg(\sigma) := f(g(\sigma))$ for any $\sigma \in \mathcal{S}$. We also let $f^2 := ff$ and, for n a positive integer, $f^n := ff^{n-1}$. We say that a configuration $\sigma \in \mathcal{S}$ is a *fixed point* for the map $f : \mathcal{S} \to \mathcal{S}$ if and only if $f(\sigma) = \sigma$. Let $\sigma \in \mathcal{S}$, consider the sequence $f^n(\sigma)$ with $n \geq 1$, if there exists n' such that $f^n(\sigma) = f^{n'}(\sigma)$ for any $n \geq n'$, we then let $\overline{f}\sigma := f^{n'}\sigma$.

2.3 The Model

Let $\beta > 0$ and $h \in \mathbb{R}$ such that $|h| < 1$ and $2/h$ is not integer. We consider the Markov chain on $\mathcal{S}$ with *transition matrix*

$$p(\sigma, \eta) := \prod_{x\in\Lambda} p_{x,\sigma}(\eta(x)) \quad \forall\sigma, \eta \in \mathcal{S} \tag{2.1}$$

where, for each $x \in \Lambda$ and $\sigma \in \mathcal{S}$, $p_{x,\sigma}(\cdot)$ is the probability measure on $\mathcal{S}_{\{x\}}$ defined as follows

$$p_{x,\sigma}(s) := \frac{1}{1 + \exp\{-2\beta s(S_\sigma(x) + h)\}} = \frac{1}{2}\left[1 + s\tanh\beta\,(S_\sigma(x) + h)\right] \tag{2.2}$$

with $s \in \{-1, +1\}$ and

$$S_\sigma(x) := \sum_{y\in\overline{\{x\}}} \sigma(y). \tag{2.3}$$

The normalization condition $p_{x,\sigma}(s) + p_{x,\sigma}(-s) = 1$ is trivially satisfied. Note that $p_{x,\cdot}(s) \in \mathcal{F}_{\overline{\{x\}}}$ for any x and s, that is the probability $p_{x,\sigma}(s)$ for the spin at site x to be equal to s depends only on the values of the five spins of σ inside the cross $\overline{\{x\}}$ centered at x.

Such a Markov chain on the finite space $\mathcal{S}$ is an example of *reversible probabilistic cellular automata* (PCA), see [6, 8]. Let $n \in \mathbb{N}$ be the discrete time variable and let $\sigma_n \in \mathcal{S}$ denote the state of the chain at time n; the configuration at time $n + 1$ is chosen according to the law $p(\sigma_n, \cdot)$, see (2.1), hence all the spins are updated simultaneously and independently at any time. Finally, given $\sigma \in \mathcal{S}$ we consider the chain with initial configuration $\sigma_0 = \sigma$, we denote with $\mathbb{P}_\sigma$ the probability measure on the space of trajectories, by $\mathbb{E}_\sigma$ the corresponding expectation value, and by

$$\tau_A^\sigma := \inf\{t > 0 : \sigma_t \in A\} \tag{2.4}$$

the *first hitting time on* $A \subset \mathcal{S}$. We shall drop the initial configuration from the notation (2.4) whenever it is equal to $-\underline{1}$, i.e., we shall write τ_A instead of $\tau_A^{-\underline{1}}$.

2.4 The Stationary Measure and the Phase Diagram

The model (2.1) has been studied numerically in [1]; we refer to that paper for a detailed discussion about its stationary properties. Here we simply recall the main features. It is straightforward, see for instance [6, 8], that the PCA (2.1) is reversible with respect to the finite volume Gibbs measure $\mu(\sigma) := \exp\{-H(\sigma)\}/Z$ with $Z := \sum_{\eta\in\mathcal{S}} \exp\{-H(\eta)\}$ and

$$H(\sigma) := H_{\beta,h}(\sigma) := -\beta h \sum_{x\in\Lambda} \sigma(x) - \sum_{x\in\Lambda} \log\cosh\big[\beta\,(S_\sigma(x)+h)\big]. \tag{2.5}$$

In other words the *detailed balance* condition

$$p(\sigma,\eta)\,e^{-H(\sigma)} = p(\eta,\sigma)\,e^{-H(\eta)} \tag{2.6}$$

is satisfied for any $\sigma,\eta\in\mathcal{S}$; hence, the measure μ is stationary for the PCA (2.1). In order to understand its most important features, it is useful to study the related Hamiltonian. Since the Hamiltonian has the form (2.5), we shall often refer to $1/\beta$ as to the *temperature* and to h as to the *magnetic field*.

The interaction is short range and it is possible to extract the potentials; following [1] we rewrite the Hamiltonian as

$$H_{\beta,h}(\sigma) = \sum_{x\in\Lambda} U_{x,\beta,h}(\sigma) - \beta h \sum_{x\in\Lambda} \sigma(x) \tag{2.7}$$

where $U_{x,\beta,h}(\sigma) = U_{0,\beta,h}(\Theta_x\sigma)$, recall that the shift operator Θ_x has been defined in Sect. 2.2 and that periodic boundary are considered on Λ, and

$$U_{0,\beta,h}(\sigma) = -\sum_{X\subset\overline{\{0\}}} J_{|X|,\beta,h} \prod_{x\in X} \sigma(x). \tag{2.8}$$

The six coefficients $J_{0,\beta,h},\ldots,J_{5,\beta,h}$ are determined by using (2.5), (2.7), and (2.8). In the case $h=0$ only even values of $|X|$ occur and we find that the pair interactions are ferromagnetic while the four-spin interactions are not. For a more detailed discussion see [1].

The definition of ground state is not completely trivial in our model, indeed the Hamiltonian H depends on β. The ground states are those configurations on which the Gibbs measure μ is concentrated when the limit $\beta\to\infty$ is considered, so that they can be defined as the minima of the *energy*

$$E(\sigma) := \lim_{\beta\to\infty} \frac{H(\sigma)}{\beta} = -h\sum_{x\in\Lambda}\sigma(x) - \sum_{x\in\Lambda} |S_\sigma(x)+h|. \tag{2.9}$$

Let $\mathcal{X}\subset\mathcal{S}$, if the energy E is constant on $\mathcal{X}$, we shall misuse the notation by denoting by $E(\mathcal{X})$ the energy of the configurations in $\mathcal{X}$.

We first consider the case $h=0$. Since $E(\sigma) = -\sum_{x\in\Lambda}|S_\sigma(x)|$, it is obvious that there exist the two minima $+\underline{1},-\underline{1}\in\mathcal{S}$, with $\pm\underline{1}(x)=\pm1$ for each $x\in\Lambda$, such that $E(+\underline{1}) = E(-\underline{1}) = -5|\Lambda|$. For $h\neq0$ we have $E(+\underline{1}) = -|\Lambda|(h+|5+h|)$ and $E(-\underline{1}) = -|\Lambda|(-h+|-5+h|)$; it is immediate to verify that $E(+\underline{1})<E(-\underline{1})$ for $h>0$ and $E(-\underline{1})<E(+\underline{1})$ for $h<0$. We conclude that at $h=0$ there exist the two ground states $-\underline{1}$ and $+\underline{1}$. At $h>0$ the unique ground state is given by $+\underline{1}$ and at $h<0$ the unique ground state is given by $-\underline{1}$. The phase diagram at finite large β and $h=0$ has been studied rigorously in [7].

2.5 Metastable Behavior

We pose now the problem of metastability and state the related theorem on the exit time. In this context, configurations with all the spins equal to minus one excepted those in rectangular subsets of the lattice will play a key role. We then let

$$\Lambda^{\pm}(\sigma) := \{x \in \Lambda : \sigma(x) = \pm 1\} \tag{2.10}$$

for any $\sigma \in \mathcal{S}$; the set $\Lambda^+(\sigma)$ will be called the *support* of σ. We say that $\sigma \in \mathcal{S}$ is a *rectangular droplet with side lengths* ℓ *and* m, with ℓ, m integers such that $2 \le \ell, m \le L-2$, if and only if there exists $x \in \Lambda$ such that either $\Lambda^+(\sigma) = Q_{\ell,m}(x)$ or $\Lambda^+(\sigma) = Q_{m,\ell}(x)$. We say that $\sigma \in \mathcal{S}$ is a *n-rectangular droplet with side lengths* $\ell_1, m_1, \dots, \ell_n, m_n$, with $n \ge 1$ an integer and ℓ_i, m_i integers such that $2 \le \ell_i, m_i \le L-2$ for $i = 1, \dots, n$, if and only if $\Lambda^+(\sigma)$ is the union of n pairwise not interacting rectangles (see Sect. 2.1) with side lengths ℓ_i and m_i for $i = 1, \dots, n$. We finally say that $\sigma \in \mathcal{S}$ is a *multi-rectangular droplet* if and only if σ is a n-rectangular droplet for some integer $n \ge 1$. Note that a 1-rectangular droplet is indeed a rectangular droplet. *Square droplets* are defined similarly.

Consider, now, the model (2.1) with $0 < h < 1$ and suppose that the system is prepared in the state $\sigma_0 = -\underline{1}$; in the infinite time limit the system tends to the phase with positive magnetization. We shall show that the minus one state is metastable in the sense that the system spends a huge amount of time close to $-\underline{1}$ before visiting $+\underline{1}$; more precisely the first hitting time $\tau_{+\underline{1}}$ to $+\underline{1}$ (recall (2.4) and the remark below) is an exponential random variable with mean exponentially large in β.

Moreover, we give some information on the exit path that the system follows during the escape from minus one to plus one. More precisely, we show that there exists a class of configurations $\mathcal{C} \subset \mathcal{S}$, called set of *critical droplets*, which is visited with high probability by the system during its escape from $-\underline{1}$ to $+\underline{1}$. Let the *critical length* λ be defined as

$$\lambda := \left\lfloor \frac{2}{h} \right\rfloor + 1 \tag{2.11}$$

where, for any positive real x, we denote by $\lfloor x \rfloor$ the *integer part* of x, i.e., the largest integer smaller than or equal to x. Since h has been chosen such that $2/h$ is not integer, see Sect. 2.3, we have that $\lambda = 2/h + \delta_h$ with $\delta_h \in (0,1)$. The set $\mathcal{C}$ is defined as the collection of configurations with all the spins equal to -1 excepted those in a rectangle of sides $\lambda - 1$ and λ and in a pair of neighboring sites adjacent to one of the longer sides of the rectangle.

Given $\gamma \in \mathcal{C}$ we let

$$\Gamma := E(\gamma) - E(-\underline{1}) + 2(1+h) = -4h\lambda^2 + 16\lambda + 4h(\lambda - 2) + 2(1+h). \tag{2.12}$$

Note that by (2.12) and (2.11) it follows

$$\Gamma < 8\lambda + 10 - 2h. \tag{2.13}$$

The simple bound above will be used in Sect. 3.2 to prove (3.25) and in Sect. 4.4.

As has been explained in the introduction, the energy of the configurations in $\mathcal{C}$ is strictly connected to the typical exit time from the metastable state, indeed we have the following theorem.

Theorem 2.1 *For $h > 0$ small enough and $L = L(h)$ large enough, we have that*

1. *for any* $\varepsilon > 0$

$$\lim_{\beta\to\infty} \mathbb{P}_{-\underline{1}}(e^{\beta\Gamma-\beta\varepsilon} < \tau_{+\underline{1}} < e^{\beta\Gamma+\beta\varepsilon}) = 1 \tag{2.14}$$

2.

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}_{-\underline{1}}[\tau_{+\underline{1}}] = \Gamma \tag{2.15}$$

3.

$$\lim_{\beta\to\infty} \mathbb{P}_{-\underline{1}}(\tau_{\mathcal{C}} < \tau_{+\underline{1}}) = 1. \tag{2.16}$$

In other words, the above theorem states that the random variable $(1/\beta)\log\tau_{+\underline{1}}$ converges in probability to Γ as $\beta \to \infty$ and that the logarithm of the mean value of $\tau_{+\underline{1}}$ divided times β converges to Γ in the same limit. Moreover, the last item ensures that, before reaching the stable state $+\underline{1}$, the system started at $-\underline{1}$ must necessarily visit the set of critical droplets $\mathcal{C}$.

The proof of Theorem 2.1 will be given in Sect. 2.8. We note that, as usual in Probabilistic Cellular Automata (see also [5]), the highest energy Γ reached along the exit path is not achieved in a configuration, which is the typical situation in Glauber dynamics. Such a Γ is the transition energy (see definition (2.18)) of the jump from the "largest subcritical" configuration to the "smallest supercritical" one, see also the heuristic discussion in Sect. 2.7.

2.6 Transition Rate and Zero Temperature Dynamics

In our problem (see also [5]) the energy difference between two configurations σ and η is not sufficient to establish whether the system prefers to jump from σ to η or vice versa. Indeed, for some pairs of configurations a sort of barrier is seen in both directions; more precisely, it is possible to find σ and η such that both $p(\sigma,\eta)$ and $p(\eta,\sigma)$ tend to zero in the zero temperature limit $\beta \to \infty$. As an example of such a behavior, consider the two following configurations: σ is such that all the spins are equal to minus one excepted those associated with the sites belonging to an $\ell \times \ell$ rectangle, with $3 \le \ell \le L-2$, and to a two-site protuberance attached to one of the sides of the rectangle; η is a configuration obtained starting from σ and flipping the spin associated with one of the sites neighboring both the rectangle and the protuberance. By using (2.1)–(2.3), it is easy to show that $p(\sigma,\eta) \sim \exp\{-2\beta(1-h)\}$ and $p(\eta,\sigma) \sim \exp\{-2\beta(1+h)\}$ for large β; see also Fig. 1, where we have reproduced the table in [1, Fig. 1] with the list of the single site event probabilities. In that figure, the large β behavior of the probability, associated to the flip of the spin at the center, is computed.

To manage those barriers we associate the *transition Hamiltonian* $H(\sigma,\eta)$ to each pair of configurations $\sigma,\eta \in \mathcal{S}$. More precisely we extend the Hamiltonian (2.5) to the function $H : \mathcal{S} \cup \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ so that

$$H(\sigma,\eta) := H(\sigma) - \log p(\sigma,\eta). \tag{2.17}$$

By the detailed balance principle (2.6), we have $H(\sigma,\eta) = H(\eta,\sigma)$ for any $\sigma,\eta \in \mathcal{S}$. Note that, by definition, $H(\sigma,\eta) \ge \max\{H(\sigma), H(\eta)\}$ and $p(\sigma,\eta) = \exp\{-[H(\sigma,\eta) - H(\sigma)]\}$; it is then reasonable to think to $H(\sigma,\eta)$ as to the Hamiltonian level reached in the transition from σ to η. As already noted in Sect. 2.4, since the Hamiltonian depends on β, it is useful to compute its limiting behavior. We then define the *transition energy*

$$E(\sigma,\eta) := \lim_{\beta\to\infty} \frac{1}{\beta} H(\sigma,\eta). \tag{2.18}$$

$$\begin{array}{ccc} & + & \\ + & + & + \\ & + & \end{array} \quad e^{-2\beta(5+h)} \qquad\qquad \begin{array}{ccc} & - & \\ - & - & - \\ & - & \end{array} \quad e^{-2\beta(5-h)}$$

$$\begin{array}{ccc} & + & \\ + & + & + \\ & - & \end{array} \quad e^{-2\beta(3+h)} \qquad\qquad \begin{array}{ccc} & - & \\ - & - & - \\ & + & \end{array} \quad e^{-2\beta(3-h)}$$

$$\begin{array}{ccc} & + & \\ + & + & - \\ & - & \end{array} \quad e^{-2\beta(1+h)} \qquad\qquad \begin{array}{ccc} & - & \\ - & - & + \\ & + & \end{array} \quad e^{-2\beta(1-h)}$$

$$\begin{array}{ccc} & - & \\ + & + & - \\ & - & \end{array} \quad 1-e^{-2\beta(1-h)} \qquad\qquad \begin{array}{ccc} & + & \\ - & - & + \\ & + & \end{array} \quad 1-e^{-2\beta(1+h)}$$

$$\begin{array}{ccc} & - & \\ - & + & - \\ & - & \end{array} \quad 1-e^{-2\beta(3-h)} \qquad\qquad \begin{array}{ccc} & + & \\ + & - & + \\ & + & \end{array} \quad 1-e^{-2\beta(3+h)}$$

Fig. 1 Large β behavior of the probabilities for the flip of the central spin for all possible configurations in the 5-spin neighborhood

Note that by using (2.9), (2.17), and the symmetry of the transition hamiltonian, we get

$$E(\sigma,\eta)=E(\eta,\sigma) \quad \text{and} \quad E(\sigma,\eta)=E(\sigma)+\Delta(\sigma,\eta)\ge \max\{E(\sigma),E(\eta)\} \tag{2.19}$$

for any $\sigma,\eta\in\mathcal{S}$, with

$$\Delta(\sigma,\eta):=-\lim_{\beta\to\infty}\frac{1}{\beta}\log p(\sigma,\eta)=\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} 2|S_\sigma(x)+h|\ge 0 \tag{2.20}$$

the *transition rate*; notice that in the second equality we have used the definition (2.9) of $E(\sigma)$, (2.17), (2.1), and (2.2).

The non-negative transition rate Δ will play a crucial role in the study of the low temperature dynamics of the model (2.1); indeed it can be proven that the model satisfies the FW condition in [12, Chapter 6], that is for any $\sigma,\eta\in\mathcal{S}$ and $\beta>0$ large enough

$$e^{-\beta\Delta(\sigma,\eta)-\beta\gamma(\beta)}\le p(\sigma,\eta)\le e^{-\beta\Delta(\sigma,\eta)+\beta\gamma(\beta)} \tag{2.21}$$

where $\gamma(\beta)$ does not depend on σ,η and tends to zero in the limit $\beta\to\infty$. From (2.21) it follows that $p(\sigma,\eta)\to 1$ for $\beta\to\infty$ if and only if $\Delta(\sigma,\eta)=0$. On the other hand, if $\Delta(\sigma,\eta)>0$, then $p(\sigma,\eta)\to 0$ exponentially fast and with rate $\Delta(\sigma,\eta)$ in the limit $\beta\to\infty$, so that Δ can be interpreted as the cost of the transition from σ to η.

To get (2.21) we first prove that for β large enough

$$\left|-\frac{1}{\beta}[H(\sigma,\eta)-H(\sigma)]+[E(\sigma,\eta)-E(\sigma)]\right|\le e^{-\beta(1-h)}. \tag{2.22}$$

The bound (2.21) shall follow easily from (2.22), (2.17), and the second equality in (2.19) relating the transition energy to the transition rate. To prove (2.22) we note that by using

(2.17), (2.1), (2.2), and (2.20) we get

$$\frac{1}{\beta}[H(\sigma,\eta)-H(\sigma)]-[E(\sigma,\eta)-E(\sigma)]=\frac{1}{\beta}\sum_{x\in\Lambda}\log(1+e^{-2\beta|S_\sigma(x)+h|}) \tag{2.23}$$

indeed,

$$\begin{aligned}
&\frac{1}{\beta}[H(\sigma,\eta)-H(\sigma)]-[E(\sigma,\eta)-E(\sigma)]\\
&\quad=\frac{1}{\beta}\sum_{x\in\Lambda}\log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h})+\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}}2\eta(x)[S_\sigma(x)+h]\\
&\quad=\frac{1}{\beta}\sum_{x\in\Lambda:\ cr\eta(x)(S_\sigma(x)+h)>0}\log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h})\\
&\qquad+\frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}}\log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h})+\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}}2\eta(x)[S_\sigma(x)+h]\\
&\quad=\frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)>0}}\log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h})+\frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}}\log(e^{+2\beta\eta(x)[S_\sigma(x)+h}+1)
\end{aligned}$$

yielding (2.23). The bound (2.22) follows once we note that $\log(1+\exp\{-2\beta|S_\sigma(x)+h|\})\geq 0$ for any $x\in\Lambda$ and $|S_\sigma(x)+h|\geq 1-h$ uniformly in $\sigma\in\mathcal{S}$ and $x\in\Lambda$, and choose $\beta\geq(\log|\Lambda|)/(1-h)$.

We finally introduce the zero temperature dynamics. Consider a configuration $\sigma\in\mathcal{S}$ and $s\in\{-1,+1\}$; since $|h|<1$, from (2.2) it follows that the probability $p_{x,\sigma}(s)$ tends either to 0 or to 1 in the limit $\beta\to\infty$. Thus, due to the product structure of (2.1), given σ there exists a unique configuration η such that $p(\sigma,\eta)\to 1$ in the limit $\beta\to\infty$. This configuration is the one such that each spin $\eta(x)$ is chosen so that $p_{x,\sigma}(\eta(x))\to 1$ for $\beta\to\infty$. We introduce the map $T:\mathcal{S}\to\mathcal{S}$, called the *zero temperature dynamics*, which associates to each $\sigma\in\mathcal{S}$ the unique configuration $T\sigma$ such that $p(\sigma,T\sigma)\to 1$ in the limit $\beta\to\infty$.

Lemma 2.2 *Given $\sigma,\eta\in\mathcal{S}$, we have that $\Delta(\sigma,\eta)=0$ if and only if $\eta=T\sigma$.*

Proof of Lemma 2.2. The lemma follows immediately by using the definition of the zero temperature dynamics T and the remarks below (2.21). □

2.7 Stable States and Stable Pairs

The proof of Theorem 2.1, although mathematically complicated, relies on a very straightforward physical argument based on a careful description of the low temperature, i.e., large β, dynamics. In this section we give an heuristic explanation of the exponential estimate (2.14) for the exit time $\tau_{+\underline{1}}$.

We introduce, first, the notion of stable configurations. If $T\sigma=\sigma$ the configuration σ is called *stable*; equivalently, we say that $\sigma\in\mathcal{S}$ is stable if and only if for any $\eta\in\mathcal{S}\setminus\{\sigma\}$ one has $p(\sigma,\eta)\to 0$ in the limit $\beta\to\infty$. If σ is not stable and $T^2\sigma=\sigma$, we say that $(\sigma,T\sigma)$ is the *stable pair* associated to σ, equivalently we say that $(\sigma,T\sigma)$ is a stable pair if and

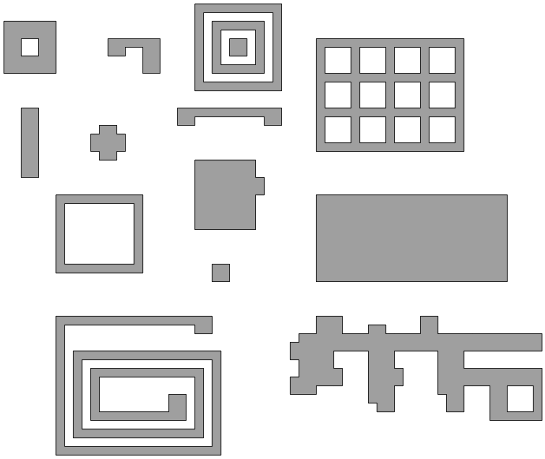

Fig. 2 Examples of stable states, pluses and minuses are represented respectively by *grey* and *white regions*

only if $p(\sigma, T\sigma) \to 1$ and $p(T\sigma, \sigma) \to 1$ in the limit $\beta \to \infty$. Recall Δ is non-negative, by (2.19) and Lemma 2.2, it follows

$$E(\sigma, T\sigma) = E(\sigma) \quad \text{and} \quad E(\sigma) \geq E(T\sigma) \tag{2.24}$$

for any $\sigma \in \mathcal{S}$.

Note that a stable pair (σ, η) is a 2-cycle of the map T, indeed $T\sigma = \eta$ and $T\eta = \sigma$. It is easy to show that cycles longer than two do not exist for such a map. Suppose, by the way of contradiction, that $\sigma_1, \dots, \sigma_n \in \mathcal{S}$, with $n \geq 3$ integer, are such that $\sigma_i \neq \sigma_j$ for $i \neq j$, $T\sigma_i = \sigma_{i+1}$ for $i = 1, \dots, n-1$, and $T\sigma_n = \sigma_1$. By the inequality in (2.24), it follows that $E(\sigma_1) \geq \dots \geq E(\sigma_n) \geq E(\sigma_1)$, which implies $E(\sigma_1) = \dots = E(\sigma_n)$. This result, together with the equality in (2.24) and (2.19), implies that $\Delta(\sigma_1, \sigma_2) = \Delta(\sigma_1, \sigma_n) = 0$. Hence, by recalling Lemma 2.2, we get $T\sigma_1 = \sigma_2$ and $T\sigma_1 = \sigma_n$. By definition of the map T, we finally get $\sigma_n = \sigma_2$, which contradicts the hypothesis that $\sigma_i \neq \sigma_j$ for $i \neq j$.

As mentioned above, our model is characterized by the presence of a large number of stable configurations. Indeed, only those configurations in which there exists at least one spin with a majority of opposite spins among its neighbors are not stable, see Fig. 1. All the configurations in which each spin is surrounded by at least two spins of the same sign are, instead, stable; some of the possible situations are shown in Fig. 2. In particular, notice that plus squared rings plunged into the sea of minuses are stable states. This scenario is complicated by the presence of stable pairs; some of them are depicted in Fig. 3. Notice, in particular, the chessboards leaned to stable pluses regions. As we shall see in the sequel, the stable pairs do not play any important role in the study of metastability in model (2.1). We also recall that, in the case of a similar model studied in [5], due to the presence of such pairs, the system was forced to visit an intermediate chessboard phase in its way from the minus metastable phase to the stable plus phase.

We describe, now, the typical low temperature behavior of the dynamics. Suppose that the initial condition is $\sigma_0 = \sigma \in \mathcal{S}$; at low temperature, with high probability, the system follows the unique *zero temperature trajectory*

$$\sigma_0 = \sigma,\ \sigma_1 = T\sigma,\ \sigma_2 = T\sigma_1 = T^2\sigma,\ \dots,\ \sigma_t = T(T^{t-1}\sigma) = T^t\sigma, \dots .$$

Once the zero temperature trajectory ends up in a stable configuration, it remains there forever. Different trajectories are observed with probability exponentially small in β.

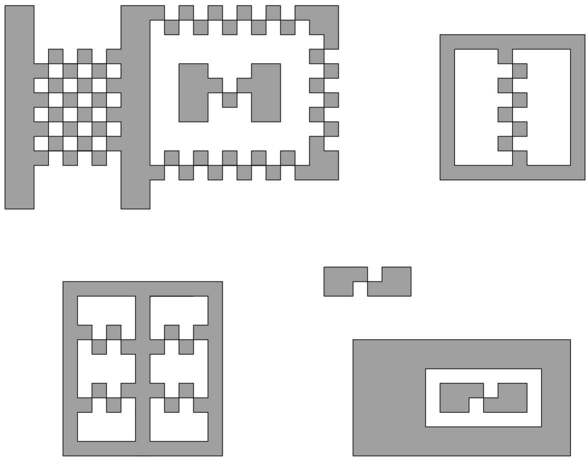

Fig. 3 Examples of stable pairs, pluses and minuses are represented respectively by *grey* and *white regions*

We can now depict the typical behavior of the system at very low temperature. Recall the definitions given in the last paragraph of Sect. 2.2. Starting from σ, the system will reach in a time of order one either the stable configuration $\overline{T}\sigma$ or the stable pair associated to $\overline{T^2}\sigma$; note that $\overline{T}\sigma$ and $\overline{T^2}\sigma$ are unique. After a time exponentially large in β, the chain will depart from the stable configuration, or from the stable pair, and possibly reach a different stable configuration, where it will remain for another exponentially long time. And so on. It is then clear that, in the study of the low temperature dynamics, a key role is played by stable configurations and stable pairs; indeed a large amount of the time of each trajectory is spent there.

Among the large number of possible stable states, there are those configurations in which the plus spins fill a rectangular region; recall the definition of rectangular droplets given at the beginning of Sect. 2.5. In [1] it has been conjectured that those rectangular stable configurations are the relevant ones for metastability. Moreover, there has been developed an heuristic argument to show that λ, see (2.11), is the *critical length* in the sense explained below. Rectangular droplets with smallest side length smaller or equal to $\lambda-1$ are *subcritical*, namely, starting from such a configuration the system visits $-\underline{1}$ before $+\underline{1}$ with probability tending to one in the limit $\beta\to\infty$. Rectangular droplets with smallest side length larger or equal to λ are *supercritical*, namely, starting from such a configuration the system visits $+\underline{1}$ before $-\underline{1}$ with probability tending to one in the limit $\beta\to\infty$.

We reproduce shortly the heuristic argument in [1, Section IV] yielding the above conclusions. Consider a square droplet of side length ℓ; we shall identify the best growth and shrinking mechanisms and, by comparing the related typical times, get the critical length. First note that the configuration obtained by attaching a single site protuberance to one of the sides of the droplet is not stable (see Fig. 1); it is needed at least a two-site protuberance to get a stable configuration. The parallel dynamics allows the formation of a two-site protuberance in one step; from Fig. 1 and the product structure of (2.1), it follows that the typical time for this process is $\tau_{\text{one}}\sim\exp\{4\beta(3-h)\}$. On the other hand, the protuberance can be formed in two consecutive steps: first a single site protuberance appears and, then, one of the two minuses adjacent both to the square droplet and to the protuberance is flipped. By using again the data in Fig. 1, we get that the typical time for the two-step process is $\tau_{\text{two}}\sim\exp\{2\beta(3-h)+4\beta(1-h)\}$, where $2\beta(3-h)$ is the cost of the first step and $4\beta(1-h)$ is the sum of the costs paid in the second step to keep the single site protuber-

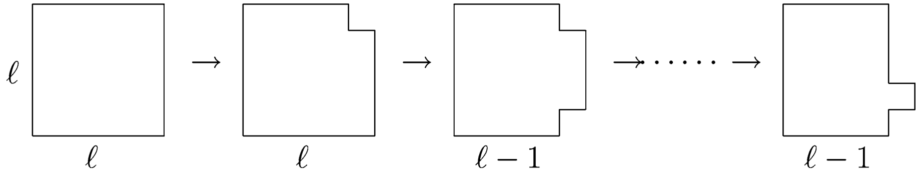

Fig. 4 Shrinking mechanism

ance and to flip the adjacent spin. Clearly $\tau_{\text{two}} \ll \tau_{\text{one}}$ for β large; hence, the most efficient mechanism to produce a two-site protuberance is the two-step one.

The presence of a two-site protuberance is sufficient to ensure the growth of the droplet. Indeed, noted that $\exp\{2\beta(1-h)\}$ is the smallest typical time needed to leave a stable configuration (see Fig. 1), we have that the side with the two-site protuberance is filled by pluses via a sequence of $\ell-2$ flips of a minus spin with two neighboring pluses. Since each of those flips happens with typical time of order $\exp\{2\beta(1-h)\}$, we conclude that the growth time τ_{growth} is equal to τ_{two}.

For what concerns the shrinking mechanism, it is easy to show that the most efficient one is the flipping of plus spins having two neighboring minuses (corner erosion). The shrinking is then performed via a sequence of configurations as in Fig. 4, requiring the erosion of $\ell-1$ corner pluses and the final flipping of the unstable single site protuberance. Note that the intermediate configurations, joining the starting $\ell\times\ell$ square droplet to the ending single protuberance configuration, are stable; their lifetime, i.e., the typical time that must be waited for to see the system performing a transition, is $\exp\{2\beta(1-h)\}$ (see Fig. 1). It follows that suitably long persistence in the $\ell-2$ intermediate stable configurations must be provided for in the most efficient shrinking path. The rate at which the entire process occurs is thus estimated as the rate for one erosion, $\exp\{-2\beta(1+h)\}$, times the probability that $\ell-2$ further erosions occur within the lifetime $\exp\{2\beta(1-h)\}$, which is of order $[\exp\{-2\beta(1+h)\}\exp\{2\beta(1-h)\}]^{\ell-2}$. We then conclude that the shrinking time is estimated as $\tau_{\text{shrinking}} \sim \exp\{2\beta(1+h)+(\ell-2)[2\beta(1+h)-2\beta(1-h)]\}$.

By comparing, finally, $\tau_{\text{shrinking}}$ and τ_{growth}, we get that growth is favored w.r.t. shrinking if and only if $\ell \ge \lfloor 2/h \rfloor + 1$. This remark strongly suggests that the length λ, defined in (2.11), plays the role of the critical length for what concerns the metastable behavior of the model.

We come, finally, to the heuristic argument suggesting the estimate (2.14) for the exit time. It is reasonable to suppose that the exit path visits an increasing sequence of subcritical rectangular droplets, whose side lengths differ at most by one. The highest energy along such a path will be attained in the segment leading from the largest subcritical $\lambda\times(\lambda-1)$ droplet to the smallest supercritical $\lambda\times\lambda$ droplet. More precisely, denote by π the configuration obtained by attaching a single site protuberance to one of the two longer sides of the $\lambda\times(\lambda-1)$ droplet and by γ the configuration obtained by flipping in π a minus spin adjacent to the rectangle and neighboring the single site protuberance. Recall the discussion above about the growth mechanism. It follows that the highest energy along the exit path must be attained in the transition from π to γ, so that it is equal to $E(\pi,\gamma)$ (see (2.18)). It is then reasonable to expect that the typical exit time is of order $\exp\{\beta[E(\pi,\gamma)-E(-\underline{1})]\}$. Using the expression

$$E(\psi)-E(-\underline{1}) = -4h\ell_1\ell_2 + 8(\ell_1+\ell_2) \tag{2.25}$$

for a rectangular droplet $\psi \in \mathcal{S}$ of side lengths ℓ_1 and ℓ_2, recall that in such a configuration $2 \le \ell_1, \ell_2 \le L-2$, it is an easy exercise to show that $E(\pi,\gamma)-E(-\underline{1}) = \Gamma$, see (2.12).

2.8 Escape Time

In this section we prove Theorem 2.1. The main ingredients will be the general results [9, Theorem 4.1, 4.9, and 5.4], the solution of the model dependent variational problem (2.29), i.e., the computation of the energy barrier between $-\underline{1}$ and $+\underline{1}$, and the recurrence estimate (2.28). In [9] the theory has been developed with quite strict hypotheses on the dynamics, see [9, equation (1.3)], nevertheless it can be shown that the same results hold in the present setup, see the Appendix.

To state the estimates on the energy landscape we need few more definitions. A finite sequence of configurations $\omega = \{\omega_1, \ldots, \omega_n\}$ is called *path* with starting configuration ω_1 and ending configuration ω_n; we let $|\omega| := n$. We let $\Omega := \mathcal{S}^{\mathbb{N}\setminus\{0\}}$ be the collection of all the possible paths. Given two paths ω and ω', such that $\omega_{|\omega|} = \omega'_1$, we let $\omega + \omega' := \{\omega_1, \ldots, \omega_{|\omega|}, \omega'_2, \ldots, \omega'_{|\omega'|}\}$; note that $|\omega + \omega'| = |\omega| + |\omega'| - 1$. Given a path ω, we define the *height* along ω as

$$\Phi_\omega := \begin{cases} E(\omega_1) & \text{if } |\omega| = 1 \\ \max_{i=1,\ldots,|\omega|-1} E(\omega_i, \omega_{i+1}) & \text{otherwise.} \end{cases} \tag{2.26}$$

Let $A, A' \subset \mathcal{S}$, we denote by $\Theta(A, A')$ the set of all the paths $\omega \in \Omega$ such that $\omega_1 \in A$ and $\omega_{|\omega|} \in A'$, that is the set of paths starting from a configuration in A and ending in a configuration in A'. The communication energy between $A, A' \subset \mathcal{S}$ is defined as

$$\Phi(A, A') := \min_{\omega \in \Theta(A, A')} \Phi_\omega. \tag{2.27}$$

If $A = \{\sigma\}$, we shall misuse the notation by writing $\Theta(\sigma, A')$ instead of $\Theta(\{\sigma\}, A')$ and $\Phi(\sigma, A')$ instead of $\Phi(\{\sigma\}, A')$.

Theorem 2.3 *Recall the definition of* Γ *in* (2.12). *Suppose that* $h > 0$ *is chosen small enough. Then*

1. *for any* $\sigma \in \mathcal{S} \setminus \{-\underline{1}\}$
$$\Phi(\sigma, +\underline{1}) - E(\sigma) < \Gamma \tag{2.28}$$
2.
$$\Phi(-\underline{1}, +\underline{1}) - E(-\underline{1}) = \Gamma \tag{2.29}$$
3. *for each path* $\omega = \{\omega_1, \ldots, \omega_n\} \in \Theta(-\underline{1}, +\underline{1})$ *such that* $\Phi_\omega - E(-\underline{1}) = \Gamma$, *there exists* $i \in \{2, \ldots, n\}$ *such that* $\omega_i \in \mathcal{C}$ *and* $E(\omega_{i-1}, \omega_i) - E(-\underline{1}) = \Gamma$.

Theorem 2.3 will be proved in Sect. 3. Recall Theorems A.1–A.3 and the definitions given before them.

Proof of Theorem 2.1. By using the results discussed at the end of Sect. 2.4, we have $\mathcal{S}^{\mathrm{s}} = \{+\underline{1}\}$. We remark that, since $\mathcal{S}^{\mathrm{s}} = \{+\underline{1}\}$, then for any $\sigma \in \mathcal{S} \setminus \mathcal{S}^{\mathrm{s}}$ we have $E(+\underline{1}) < E(\sigma)$; this implies, together with (2.29) and (2.28), that $\mathcal{S}^{\mathrm{m}} = \{-\underline{1}\}$ and $V_{-\underline{1}} = \Gamma$. Finally, items 1 and 2 follow from Theorems A.1 and A.2, respectively.

Proof of item 3. By using item 3 in Theorem 2.3, we get that $\mathcal{C}$ is a gate for the transition from $-\underline{1}$ to $+\underline{1}$. The item 3 in Theorem 2.1 follows from Theorem A.3. □

3 The Recurrence Property and the Variational Problem

In this section we prove the energy landscape estimates stated in Theorem 2.3; in particular the recurrence property (2.28) is proven in Sect. 3.1 and the variational problem (2.29) is solved in Sect. 3.2. The proof of items 2 and 3 of Theorem 2.3 relies on Proposition 3.2, which is stated in Sect. 3.2 and proven in Sect. 4.

We give few more definitions. Let $\sigma \in \mathcal{S}$ and $x \in \Lambda$, we say that the site x is *stable* (resp. *unstable*) w.r.t. σ if and only if $\sigma(x)S_\sigma(x) > 0$ (resp. $\sigma(x)S_\sigma(x) < 0$). Note that the stable sites are those that are not changed by the zero temperature dynamics, more precisely $T\sigma(x) = \sigma(x)$ if and only if x is stable w.r.t. σ. Given $\sigma \in \mathcal{S}$ and $k \in \{-5, -3, -1, +1, +3, +5\}$ we denote by $\Lambda_k^\pm(\sigma)$ the collection of the sites $x \in \Lambda^\pm(\sigma)$ such that $S_\sigma(x) = k$, i.e.,

$$\Lambda_k^\pm(\sigma) := \{x \in \Lambda^\pm(\sigma) : S_\sigma(x) = k\} \tag{3.1}$$

note that $\Lambda_{-5}^+(\sigma) = \emptyset$ and $\Lambda_{+5}^-(\sigma) = \emptyset$; moreover, we set

$$\Lambda_{\le k}^\pm(\sigma) := \Lambda_{-5}^\pm(\sigma) \cup \cdots \cup \Lambda_k^\pm(\sigma) \quad \text{and} \quad \Lambda_{\ge k}^\pm(\sigma) := \Lambda_k^\pm(\sigma) \cup \cdots \cup \Lambda_{+5}^\pm(\sigma). \tag{3.2}$$

Finally, given $\sigma \in \mathcal{S}$, we denote by $\Lambda_{\mathrm{s}}^+(\sigma)$ (resp. $\Lambda_{\mathrm{u}}^+(\sigma)$) the collection of the sites $x \in \Lambda$ such that $\sigma(x) = +1$ and x is stable (resp. unstable) w.r.t. σ; similarly we define $\Lambda_{\mathrm{s}}^-(\sigma)$ and $\Lambda_{\mathrm{u}}^-(\sigma)$. By definition of stable and unstable sites we get that, for any $\sigma \in \mathcal{S}$,

$$\begin{aligned}
&\Lambda_{\mathrm{u}}^+(\sigma) = \Lambda_{\le -1}^+(\sigma), \quad \Lambda_{\mathrm{u}}^-(\sigma) = \Lambda_{\ge +1}^-(\sigma), \quad \Lambda_{\mathrm{s}}^+(\sigma) = \Lambda_{\ge +1}^+(\sigma), \quad \text{and}\\
&\Lambda_{\mathrm{s}}^-(\sigma) = \Lambda_{\le -1}^-(\sigma).
\end{aligned} \tag{3.3}$$

3.1 The Recurrence Property

Equation (2.28) in Theorem 2.3 states that, for any configuration σ different from the metastable state $-\underline{1}$, it is possible to exhibit a path ω joining σ to the stable state $+\underline{1}$, i.e., to the absolute minimum of the energy, such that $\Phi_\omega < E(\sigma) + \Gamma$. On the heuristic ground, given $\sigma \in \mathcal{S} \setminus \{-\underline{1}\}$, there exists at least a plus spin; starting from such a plus it is possible to build a supercritical $\lambda \times \lambda$ droplet of pluses paying an energy cost strictly smaller than $E(\sigma) + \Gamma$. Indeed, by virtue of (2.29), starting from $-\underline{1}$, the cost would be exactly Γ. On the other hand, starting from σ, no energy must be paid to get the first plus spin and the other pluses of σ, if any, help the production of the supercritical droplet.

A rigorous proof needs the explicit construction of the path; such a path will firstly realize the growth of a supercritical $\lambda \times \lambda$ square with σ as a background and then its growth towards $+\underline{1}$. More precisely, recall Λ is a squared torus, let L be its side length and $0 = (0, 0)$ the origin; recall the zero temperature dynamics mapping T defined in Sect. 2.6 and let $\sigma \in \mathcal{S}$ be such that $\sigma(x) = +1$ for any $x \in Q_{2,2}(0)$. We define the path

$$\Omega_\sigma := \Xi^2 + \sum_{n=3}^{L} [\Psi^n + \Xi^n] \tag{3.4}$$

where the paths Ξ^n, with $n = 2, \ldots, L$, and Ψ^n, with $n = 3, \ldots, L$, are constructed algorithmically.

We first describe informally the algorithms. The path Ξ^n starts from the configuration ξ^n and ends in the configuration ψ^{n+1}. The configuration ξ^n is such that the square $Q_{n,n}(0) = \{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$ is filled with pluses; the path Ξ^n fills with pluses the slice

$Q_{1,n}(n,0)=\{(n,0),\dots,(n,n-1)\}$, adjacent to the square $Q_{n,n}(0)$, and produce ψ^{n+1} in which the rectangle $Q_{n+1,n}(0)$ is filled with pluses. Similarly, the path Ψ^n starts from the configuration ψ^n and ends in the configuration ξ^n. The configuration ψ^n is such that the rectangle $Q_{n,n-1}(0)=\{0,\dots,n-1\}\times\{0,\dots,n-2\}$ is filled with pluses; the path Ψ^n fills with pluses the slice $Q_{n,1}(0,n-1)=\{(0,n-1),\dots,(n-1,n-1)\}$, adjacent to the rectangle $Q_{n,n-1}(0)$, and produce ξ^n in which the square $Q_{n,n}(0)$ is filled with pluses.

Definition of Ξ^n. Let $\xi^2:=\sigma$, let $n\in\{2,\dots,L-1\}$, and suppose ξ^n is such that $\xi^n(x)=+1$ for $x\in Q_{n,n}(0)$, then

1. `set` $i=1$, $\xi_i^n=\xi^n$;
2. `if` $T^2\xi_i^n=\xi_i^n$ `then goto` 3 `else set` $i=i+1$ `and` $\xi_i^n=T\xi_{i-1}^n$ `and goto` 2;
3. `if` $\xi_i^n(x)=+1$ for all $x\in Q_{1,n}(n,0)$ `then set` $\psi^{n+1}=\xi_i^n$ `and goto` 7;
4. `if` $Q_{1,n}(n,0)\cap\Lambda_s^+(\xi_i^n)\neq\emptyset$, `then pick` $y,y'\in Q_{1,n}(n,0)$ such that $d(y,y')=1$, $\xi_i^n(y)=-1$, and $y'\in\Lambda_s^+(\xi_i^n)$, `set` $i=i+1$, $\xi_i^n(y)=+1$, $\xi_i^n(x)=T\xi_{i-1}^n(x)$ $\forall x\in\Lambda\setminus\{y\}$ `and goto` 3;
5. `if` $Q_{1,n}(n,0)\cap\Lambda_u^+(\xi_i^n)\neq\emptyset$, `then pick` $y,y'\in Q_{1,n}(n,0)$ such that $d(y,y')=1$, $\xi_i^n(y)=-1$, and $y'\in\Lambda_u^+(\xi_i^n)$, `set` $i=i+1$, $\xi_i^n(y)=+1$, $\xi_i^n(y')=+1$, $\xi_i^n(x)=T\xi_{i-1}^n(x)$ for any $x\in\Lambda\setminus\{y,y'\}$ `and goto` 3;
6. `set` $i=i+1$, $y=(n,0)$, $\xi_i^n(y)=+1$, $\xi_i^n(x)=T\xi_{i-1}^n(x)$ for any $x\in\Lambda\setminus\{y\}$ `and goto` 3;
7. `set` $h_n=i$, $\Xi^n=\{\xi_1^n,\dots,\xi_{h_n}^n\}$ `and exit`.

At step 2 the algorithm follows the zero temperature dynamics down to the stable pair or to the stable state associated to ξ^n. At step 3 the algorithm checks if the slice $Q_{1,n}(n,0)$ adjacent to the square $Q_{n,n}(0)$ is filled with pluses. In case of positive answer the algorithm jumps to step 7 and exits. If the answer is negative, then the slice is filled with pluses at steps 3–6 as follows: a minus adjacent to a stable plus is flipped (step 4); in absence of stable pluses, a minus adjacent to an unstable plus is flipped (step 5). If the slice is filled with minuses, then the spin associated to the site $(n,0)$ is flipped (step 6).

Definition of Ψ^n. Let $n\in\{3,\dots,L\}$ and suppose ψ^n is such that $\psi^n(x)=+1$ for $x\in Q_{n,n-1}(0)$, then

1. `set` $i=1$, $\psi_i^n=\psi^n$;
2. `if` $T^2\psi_i^n=\psi_i^n$ `then goto` 3 `else set` $i=i+1$ `and` $\psi_i^n=T\psi_{i-1}^n$ `and goto` 2;
3. `if` $\psi_i^n(x)=+1$ for all $x\in Q_{n,1}(0,n-1)$ `then set` $\xi^n=\psi_i^n$ `and goto` 7;
4. `if` $Q_{n,1}(0,n-1)\cap\Lambda_s^+(\psi_i^n)\neq\emptyset$, `then pick` $y,y'\in Q_{n,1}(0,n-1)$ such that $d(y,y')=1$, $\psi_i^n(y)=-1$, and $y'\in\Lambda_s^+(\psi_i^n)$, `set` $i=i+1$, $\psi_i^n(y)=+1$, $\psi_i^n(x)=T\psi_{i-1}^n(x)$ $\forall x\in\Lambda\setminus\{y\}$ `and goto` 3;
5. `if` $Q_{n,1}(0,n-1)\cap\Lambda_u^+(\psi_i^n)\neq\emptyset$, `then pick` $y,y'\in Q_{n,1}(0,n-1)$ such that $d(y,y')=1$, $\psi_i^n(y)=-1$, and $y'\in\Lambda_u^+(\psi_i^n)$, `set` $i=i+1$, $\psi_i^n(y)=+1$, $\psi_i^n(y')=+1$, $\psi_i^n(x)=T\psi_{i-1}^n(x)$ for any $x\in\Lambda\setminus\{y,y'\}$ `and goto` 3
6. `set` $i=i+1$, $y=(0,n)$, $\psi_i^n(y)=+1$, $\psi_i^n(x)=T\psi_{i-1}^n(x)$ for any $x\in\Lambda\setminus\{y\}$ `and goto` 3;
7. `set` $k_n=i$, $\Psi^n=\{\psi_1^n,\dots,\psi_{k_n}^n\}$ `and exit`.

In the following lemma we summarize the main properties of the paths Ξ^n and Ψ^n defined above. In particular in items 2 and 3 we state upper bounds on the energy levels reached by those paths. We show that the addition of a slice of pluses to a rectangle of pluses can result in a net increment of the energy only if the length of the added slice does not exceed the critical length λ (see (2.11), (3.5), and (3.6)).

Lemma 3.1 *Let $\sigma \in \mathcal{S}$ be such that $\sigma(x) = +1$ for any $x \in Q_{2,2}(0)$, consider the path Ω_σ defined by (3.4). Then*

1. *for any $n = 3, \dots, L$ the configuration ψ^n is such that $\psi^n(x) = +1$ for all $x \in Q_{n,n-1}(0)$, for any $n = 3, \dots, L$ the configuration ξ^n is such that $\xi^n(x) = +1$ for all $x \in Q_{n,n}(0)$, in particular $\xi^L = +\underline{1}$ and $\Xi^L = \{\xi^L\}$;*
2. *for any $n = 2, \dots, L$ we have*
$$E(\psi^{n+1}) - E(\xi^n) \le (8 - 4hn) \vee 0 \quad \text{and} \quad \Phi_{\Xi^n} \le E(\xi^n) + 10 - 6h \tag{3.5}$$
3. *for any $n = 3, \dots, L$ we have*
$$E(\xi^n) - E(\psi^n) \le (8 - 4hn) \vee 0 \quad \text{and} \quad \Phi_{\Psi^n} \le E(\psi^n) + 10 - 6h \tag{3.6}$$
4. *we have*
$$\Phi_{\Omega_\sigma} - E(\sigma) \le \Gamma - 16(2 - h) \tag{3.7}$$
where we recall Γ has been defined in (2.12).

Proof of Lemma 3.1. Item 1 is an immediate consequence of the algorithmic definition of Ω_σ. The proof of item 2 is similar to the proof of item 3.

Item 3. Let $\Psi^n := \{\psi_1^n, \dots, \psi_k^n, \dots, \psi_{k_n}^n\}$, with $k_n \ge k \ge 1$, such that $\psi_i^n = T\psi_{i-1}^n$ for $i = 2, \dots, k$ and $\psi_k^n = T^2\psi_k^n$; note that by construction $\psi_1^n = \psi^n$, $\psi_{k_n}^n = \xi^n$, and $k_n - k \le n$. By using (2.24), we get
$$\Phi_{\{\psi_1^n, \dots, \psi_k^n\}} = E(\psi_1^n) \quad \text{and} \quad E(\psi_i^n) \ge E(\psi_{i+1}^n) \tag{3.8}$$
for $i = 1 \dots, k - 1$. If $k_n = k$, then (3.6) follows immediately from (3.8). In the case $k_n \ge k + 1$, we shall prove that
$$\Phi_{\{\psi_k^n, \psi_{k+1}^n, \dots, \psi_{k_n}^n\}} \le E(\psi_k^n) + 10 - 6h \quad \text{and} \quad E(\psi_{k_n}^n) - E(\psi_k^n) \le (8 - 4hn) \vee 0. \tag{3.9}$$
The bounds (3.6) will then follow from (3.8) and (3.9).

We are then left with the proof of (3.9), which can be achieved by discussing the following three cases.

Case 1. There exist $y, y' \in Q_{n,1}(0, n - 1)$ such that $\psi_k^n(y) = -1$, $y' \in \Lambda_s^+(\psi_k^n)$. The configuration ψ_{k+1}^n is defined at the step 4 of the algorithm; it is immediate to see that all the configurations ψ_i^n, with $i = k + 1, \dots, k_n$, are indeed defined at the step 4. Then, by using (2.18), see also Fig. 1, we get the following bounds on the transition energies:
$$E(\psi_i^n, \psi_{i+1}^n) \le E(\psi_i^n) + 2(1 - h) \quad \text{and} \quad E(\psi_{i+1}^n, \psi_i^n) \ge E(\psi_{i+1}^n) + 2(1 + h) \tag{3.10}$$
for any $i = k, \dots, k_n - 1$. By using (3.10), (2.26), and (2.19) we get
$$\Phi_{\{\psi_k^n, \psi_{k+1}^n, \dots, \psi_{k_n}^n\}} \le E(\psi_k^n) + 2(1 - h) \quad \text{and} \quad E(\psi_{k_n}^n) - E(\psi_k^n) \le -4h(k_n - k) \tag{3.11}$$
which, recalling $k_n \ge k + 1$, imply (3.9).

Case 2. There exist $y, y' \in Q_{n,1}(0, n - 1)$ such that $\psi_k^n(y) = -1$, $\psi_k^n(y') = +1$, and $\Lambda_s^+(\psi_k^n) \cap Q_{n,1}(0, n - 1) = \emptyset$. The configuration ψ_{k+1}^n is defined at the step 5 of the algorithm; it is immediate to remark that all the configurations ψ_i^n, with $i = k + 1, \dots, k_n$, are instead defined at the step 4.

Let $i = k, \dots, k_n - 1$, let $y, y' \in Q_{n,1}(0, n-1)$ be the two sites which are picked up by the algorithm, let Δ_i be the collection of the sites in $Q_{n,1}(0, n-1)$ different from y, y' and such that they become stable plus sites at this step of the path; more precisely, let $\Delta_i := \Lambda_s^+(\psi_{i+1}^n) \setminus (\Lambda_s^+(\psi_i^n) \cup \{y, y'\})$. Note that the update of the sites in Δ_i has no energy cost since they follow the zero temperature dynamics T.

By using (2.18), see also Fig. 1, we get the estimates

$$\begin{aligned} &E(\psi_k^n, \psi_{k+1}^n) \le E(\psi_k^n) + 4(1-h) \quad E(\psi_{k+1}^n, \psi_k^n) \ge E(\psi_{k+1}^n) + 2(1+h)(1+|\Delta_k|) \\ &E(\psi_i^n, \psi_{i+1}^n) \le E(\psi_i^n) + 2(1-h) \quad E(\psi_{i+1}^n, \psi_i^n) \ge E(\psi_{i+1}^n) + 2(1+h)(1+|\Delta_i|) \end{aligned} \tag{3.12}$$

for any $i = k+1, \dots, k_n - 1$. If $|\Delta_i| = 0$ for any $i = k, \dots, k_n - 1$, then it must necessarily be $k_n - k = n - 1$. We get

$$\Phi_{\{\psi_k^n, \psi_{k+1}^n, \dots, \psi_{k_n}^n\}} \le E(\psi_k^n) + 4(1-h) \quad \text{and} \quad E(\psi_{k_n}^n) - E(\psi_k^n) \le 2 - 2h - 4h(n-1). \tag{3.13}$$

Noted that $8 - 4hn = 2 - 2h - 4h(n-1) + (6 - 2h)$, the bound (3.9) follows since $h \le 3$. Suppose, finally, that there exists $i \in \{k, \dots, k_n - 1\}$ such that $|\Delta_i| \ne 0$; hence

$$\Phi_{\{\psi_k^n, \psi_{k+1}^n, \dots, \psi_{k_n}^n\}} \le E(\psi_k^n) + 4(1-h) \quad \text{and} \quad E(\psi_{k_n}^n) - E(\psi_k^n) \le -4h(k_n - k + 1). \tag{3.14}$$

Recall $k_n \ge k + 1$; the bounds (3.14) imply (3.9) trivially.

Case 3. For each $y \in Q_{n,1}(0, n-1)$ we have $\psi_k^n(y) = -1$. In this case $k_n - k = n$, ψ_{k+1}^n is defined at the step 6, ψ_{k+2}^n is defined either at the step 4 or at the step 5, and ψ_{k+i}^n, with $i = 3, \dots, k_n$, are defined at the step 4 of the algorithm. By using (2.18), see also Fig. 1, we get

$$\begin{aligned} &E(\psi_k^n, \psi_{k+1}^n) \le E(\psi_k^n) + 2(3-h) && E(\psi_{k+1}^n, \psi_k^n) \ge E(\psi_{k+1}^n) \\ &E(\psi_{k+1}^n, \psi_{k+2}^n) \le E(\psi_{k+1}^n) + 4(1-h) && E(\psi_{k+2}^n, \psi_{k+1}^n) \ge E(\psi_{k+2}^n) + 2(1+h) \\ &E(\psi_i^n, \psi_{i+1}^n) \le E(\psi_i^n) + 2(1-h) && E(\psi_{i+1}^n, \psi_i^n) \ge E(\psi_{i+1}^n) + 2(1+h) \end{aligned} \tag{3.15}$$

for $i = k+2, \dots, k_n - 1$; see Fig. 5 for a graphical representation of the estimates (3.15). Note that the equalities hold, for instance, in the case $\psi_k^n(x) = -1$ for any $x \in \partial Q_{n,1}(0, n -$

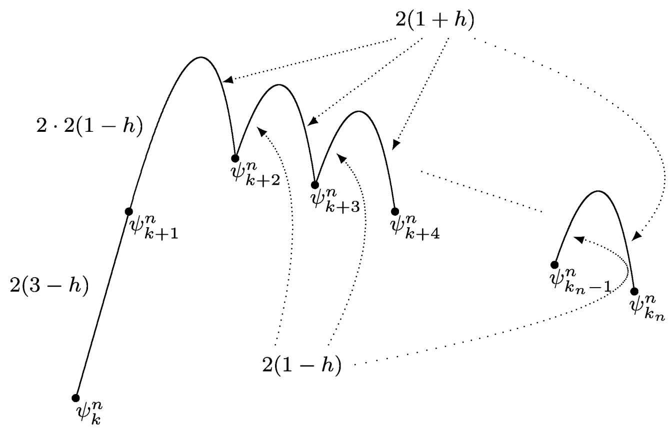

Fig. 5 Graphical representation of the estimates (3.15)

$1) \setminus Q_{n,n-1}(0)$. By using (3.15), (2.26), and (2.19) we get

$$\Phi_{\{\psi^n_k,\psi^n_{k+1},\dots,\psi^n_{k_n}\}} \le E(\psi^n_k)+10-6h \quad \text{and} \quad E(\psi^n_{k_n})-E(\psi^n_k) \le [8-4h(k_n-k)] = 8-4hn \tag{3.16}$$

which imply (3.9).

We remark that in this case 3 the path $\{\psi^n_k,\dots,\psi^n_{k_n}\}$ realizes the standard growth of the rectangular plus droplet ψ^n up to the square droplet $\psi^n_{k_n}$ via the formation of a unit plus protuberance in the slice adjacent to one of the longer sides of the rectangle and the bootstrap percolation plus filling of the same slice.

Item 4. Let η,η' two consecutive configurations of the path Ω_σ, we shall prove that

$$E(\eta,\eta')-E(\sigma) \le \Gamma - 16(2-h). \tag{3.17}$$

The bound (3.7) will then follow, see (2.26). Recall the critical length λ has been defined in (2.11) and consider the following four cases.

Case 1. The configurations η,η' belong to Ξ^n for some $n\le\lambda-1$. This case is similar to the case 2.

Case 2. The configurations η,η' belong to Ψ^n for some $n\le\lambda$. By using (2.26), (3.5), and (3.6) we have

$$\begin{aligned}
E(\eta,\eta') &\le \Phi_{\Psi^n} \le E(\psi^n)+10-6h \\
&\le E(\psi^n)-E(\xi^{n-1})+E(\xi^{n-1})-\dots-E(\psi^3)+E(\psi^3)-E(\xi^2)+E(\xi^2) \\
&\quad +10-6h \\
&\le E(\sigma)+18-14h+8\sum_{i=3}^{n-1}[2-hi] \le E(\sigma)+18-14h+8\sum_{i=3}^{\lambda-1}[2-hi]
\end{aligned}$$

where we have used that $2-hi>0$ for $i\le\lambda-1$ and $\xi^2=\sigma$. The bound (3.17) follows easily.

Case 3. The configurations η,η' belong to Ξ^n for some $n\ge\lambda$. Note that for $n\ge\lambda$ the bounds (3.5) and (3.6) on the differences of energy become trivial since $8-4hn<0$, hence $E(\xi^n)\le E(\psi^\lambda)$. Then

$$E(\eta,\eta') \le \Phi_{\Xi^n} \le E(\xi^n)+10-6h \le E(\psi^\lambda)+10-6h$$

where in the first inequality we used (2.26), in the second the bound (3.5), and in the last the fact that $E(\xi^n)\le E(\psi^\lambda)$. To get (3.17) we then perform the same computation as in the case 2.

Case 4. The configurations η,η' belong to Ψ^n for some $n\ge\lambda+1$. This case is similar to the case 3. $\square$

Proof of item 1 of Theorem 2.3. Let $\sigma\in\mathcal{S}\setminus\{-\underline{1}\}$. If $\sigma=+\underline{1}$ the statement of the lemma is trivial; we then suppose $\sigma\ne+\underline{1}$. Since by hypothesis $\sigma\ne-\underline{1}$, there exists $x\in\Lambda$ such that $\sigma(x)=+1$; without loss of generality we suppose $\sigma(0)=+1$. Consider the path $\omega:=\{\sigma,\sigma^1,\sigma^2,\sigma^3\}$ with

– σ^1 is such that $\sigma^1(x)=+1$ for all $x\in Q_{2,1}(0)$ and $\sigma^1(x)=T\sigma(x)$ for all $x\in\Lambda\setminus Q_{2,1}(0)$;

– σ^2 is such that $\sigma^2(x) = +1$ for all $x \in Q_{2,1}(0) \cup Q_1(0,1)$ and $\sigma^2(x) = T\sigma^1(x)$ for all $x \in \Lambda \setminus [Q_{2,1}(0) \cup Q_1(0,1)]$;
– σ^3 is such that $\sigma^3(x) = +1$ for all $x \in Q_{2,2}(0)$ and $\sigma^3(x) = T\sigma^2(x)$ for all $x \in \Lambda \setminus Q_{2,2}(0)$.

By definition the path $\omega + \Omega_{\sigma^3}$ starts at σ and ends in $+\underline{1}$, i.e., $\omega + \Omega_{\sigma^3} \in \Theta(\sigma, +\underline{1})$, moreover we shall prove that

$$\Phi_{\omega+\Omega_{\sigma^3}} < E(\sigma) + \Gamma. \tag{3.18}$$

Item 1 of Theorem 2.3 will then follow.

To prove (3.18) we first consider the path ω; by using (2.18), see also Fig. 1, we get the following bounds on the transition energies:

$$\begin{array}{ll} E(\sigma,\sigma^1) \le E(\sigma) + 2\cdot 2(3-h) & E(\sigma^1,\sigma) \ge E(\sigma^1) \\ E(\sigma^1,\sigma^2) \le E(\sigma^1) + 2\cdot 2(1-h) + 2(3-h) & E(\sigma^2,\sigma^1) \ge E(\sigma^2) \\ E(\sigma^2,\sigma^3) \le E(\sigma^2) + 3\cdot 2(1-h) & E(\sigma^3,\sigma^2) \ge E(\sigma^3) + 2(1+h). \end{array} \tag{3.19}$$

By using (3.19), (2.26), (2.19), (2.12), and the definition (2.11) of the critical length λ, it is easy to show that

$$\Phi_\omega - E(\sigma) \le 28 - 16h < \Gamma \tag{3.20}$$

and

$$E(\sigma^3) - E(\sigma) \le 26 - 18h. \tag{3.21}$$

We consider, now, the path Ω_{σ^3}; by using (3.7) and (3.21), we get

$$\Phi_{\Omega_{\sigma^3}} - E(\sigma) = \Phi_{\Omega_{\sigma^3}} - E(\sigma^3) + E(\sigma^3) - E(\sigma) \le \Gamma - 16(2-h) + 26 - 18h = \Gamma - 2(3+h). \tag{3.22}$$

The inequality (3.18) follows from (3.20) and (3.22). □

3.2 The Variational Problem

Item 2 of Theorem 2.3 deals with the determination of the minimal energy barrier between the metastable state $-\underline{1}$ and the stable one $+\underline{1}$, more precisely with the computation of $\Phi(-\underline{1}, +\underline{1})$. In the context of serial Glauber dynamics this problem has been faced with different approaches each suited to the model under exam, see [12] and [9, Section 4.2], where a quite general technique is described. All these methods rely on the continuity of the dynamics, namely, on the property that at each step only one spin is updated.

In the case of parallel dynamics, see [5], the lacking of continuity increases the difficulty of the computation of the communication energy between the metastable and the stable state. We follow, here, the method proposed in [5] which is based on the construction of a set $\mathcal{G} \subset \mathcal{S}$ containing $-\underline{1}$, but not $+\underline{1}$, and on the evaluation of the transition energy for all the possible transitions from the interior to the exterior of such a set $\mathcal{G}$.

To define the set $\mathcal{G}$ we need to introduce the two mappings $A, B : \mathcal{S} \to \mathcal{S}$. Let $\sigma \in \mathcal{S}$, we set $A\sigma := \sigma$ if $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_u^+(\sigma)$, otherwise $A\sigma := \sigma^x$ where x is the first element of $\Lambda_u^+(\sigma)$ in lexicographic order. The map A flips the first, in lexicographic order, unstable plus spin of σ to which corresponds a decrease of the energy. Under the effect of the map A the number of pluses decreases, but only unstable pluses are flipped. Let $\sigma \in \mathcal{S}$, the configuration $B\sigma \in \mathcal{S}$ is such that for each $x \in \Lambda$

$$B\sigma(x) := \begin{cases} -\sigma(x) & \text{if } x \in \Lambda^-_{\geq -1}(\sigma) \\ \sigma(x) & \text{otherwise.} \end{cases} \tag{3.23}$$

Note that the operator B performs a single step of *bootstrap percolation*; relatively to σ, it flips all the minus unstable spins and, among the stable minus spins, only those with two neighboring minuses.

In the sequel a relevant role will be played by the configuration $\overline{B}\,\overline{A}\sigma$, for any $\sigma \in \mathcal{S}$; recall the definition of fixed point of a map given at the end of Sect. 2.2. The sole unstable positive spins in $\overline{A}\sigma$ are those corresponding to energy increasing flips. Starting from $\overline{A}\sigma$, the map B, which flips the minus spins with at least two plus spins among the nearest neighbors, is applied iteratively until a fixed point is reached. It is easy to show that the pluses in such a fixed point form well separated rectangles or stripes winding around the torus; more precisely, the pluses in $\overline{B}\,\overline{A}\sigma$ occupy the region $\bigcup_{i=1}^{n} Q_{\ell_{i,1},\ell_{i,2}}(x_i) \subset \Lambda$, where $n, \ell_{1,1}, \ell_{1,2}, \ldots, \ell_{n,1}, \ell_{n,2}$ are positive integers and $x_i \in \Lambda$ for any $i = 1, \ldots, n$, with $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ being pairwise not interacting (see Sect. 2.1). Note that, depending on the values of $\ell_{i,1}, \ell_{i,2}$, the set $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ can be either a rectangle or a stripe winding around the torus.

We can now define the set $\mathcal{G}$. Let $\sigma \in \mathcal{S}$, consider $\overline{B}\,\overline{A}\sigma$, and, provided $\overline{B}\,\overline{A}\sigma \neq -\underline{1}$, denote by $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ the collection of pairwise not interacting rectangles (or stripes) obtained by collecting all the sites $y \in \Lambda$ such that $\overline{B}\,\overline{A}\sigma(y) = +1$. We say that $\sigma \in \mathcal{G}$ if and only if $\overline{B}\,\overline{A}\sigma = -\underline{1}$ or $\min\{\ell_{i,1}, \ell_{i,2}\} \leq \lambda - 1$ and $\max\{\ell_{i,1}, \ell_{i,2}\} \leq L - 2$ for any $i = 1, \ldots, n$. Note that configurations σ such that $\overline{B}\,\overline{A}\sigma$ contains plus stripes winding around the torus Λ do not belong to $\mathcal{G}$.

In general $\overline{T}\sigma \neq \overline{B}\,\overline{A}\sigma$, this means that $\overline{B}\,\overline{A}\sigma$ is not necessarily the result of the zero temperature dynamics started at σ. This is not a problem when looking for the minimal energy barrier between $-\underline{1}$ and $+\underline{1}$, provided the energy of such configurations is larger than Γ. The definition of $\mathcal{G}$ is indeed satisfactory because we can prove the following Proposition 3.2 on which the proof of items 2 and 3 of Theorem 2.3 is mostly based. To state the lemma we need one more definition: recall the set $\mathcal{C}$ is defined as the collection of configurations with all the spins equal to -1 excepted those in a rectangle of sides $\lambda - 1$ and λ and in a pair of neighboring sites adjacent to one of the longer sides of the rectangle. Then, given $\gamma \in \mathcal{C}$, we let $\pi(\gamma) \subset \mathcal{S}$ the set whose elements are the two configurations that can be obtained from γ by flipping one of the two plus spins in the pair attached to one of the longer sides of the plus spin $\lambda \times (\lambda - 1)$ rectangle. We also let $\mathcal{P}$ be the collection of all the configurations with all the spins equal to -1 excepted those in a rectangle of sides $\lambda - 1$ and λ and in a single site adjacent to one of the longer sides of the rectangle. Finally, we let $\mathcal{R}$ be the collection of rectangular droplets with sides $\lambda - 1$ and λ. By using (2.25), we have

$$E(\mathcal{R}) - E(-\underline{1}) = -4h\lambda^2 + 4h\lambda + 16\lambda - 8 = \Gamma - 10 + 6h \tag{3.24}$$

where we have used in the last equality the definition (2.12) of Γ. By using (2.13), we have the easy bound

$$E(\mathcal{R}) - E(-\underline{1}) < 8\lambda + 4h \tag{3.25}$$

Proposition 3.2 *With the definitions above, for $h > 0$ small enough and $L = L(h)$ large enough, we have*

1. $-\underline{1} \in \mathcal{G}$, $+\underline{1} \in \mathcal{S} \setminus \mathcal{G}$, *and* $\mathcal{C} \subset \mathcal{S} \setminus \mathcal{G}$;
2. *for each* $\eta \in \mathcal{G}$ *and* $\zeta \in \mathcal{S} \setminus \mathcal{G}$ *we have* $E(\eta, \zeta) \geq E(-\underline{1}) + \Gamma$;
3. *for each* $\eta \in \mathcal{G}$ *and* $\zeta \in \mathcal{S} \setminus \mathcal{G}$ *we have* $E(\eta, \zeta) = E(-\underline{1}) + \Gamma$ *if and only if* $\zeta \in \mathcal{C}$ *and* $\eta \in \pi(\zeta)$.

Proof of item 2 of Theorem 2.3. Since $-\underline{1} \in \mathcal{G}$ and $+\underline{1} \in \mathcal{S} \setminus \mathcal{G}$, see item 1 in Proposition 3.2, we have that any path $\omega = \{\omega_1, \ldots, \omega_n\}$ such that $\omega_1 = -\underline{1}$ and $\omega_n = +\underline{1}$ must necessarily contain a transition from $\mathcal{G}$ to $\mathcal{S} \setminus \mathcal{G}$, i.e., there must be $i \in \{2, \ldots, n\}$ such that $\omega_{i-1} \in \mathcal{G}$ and $\omega_i \in \mathcal{S} \setminus \mathcal{G}$. Thus, item 2 in Proposition 3.2 implies that $\Phi_\omega \geq E(-\underline{1}) + \Gamma$; since the path ω is arbitrary, it follows that

$$\Phi(-\underline{1}, +\underline{1}) \geq E(-\underline{1}) + \Gamma. \tag{3.26}$$

To complete the proof of (2.29) we need to exhibit a path connecting $-\underline{1}$ to $+\underline{1}$ such that the height along such a path is less than or equal to $E(-\underline{1}) + \Gamma$. Consider the path $\omega := \{-\underline{1}, \sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ with σ^1 the configuration with all the spins equal to minus one excepted the one at the origin, σ^2 the configuration with all the spins equal to minus one excepted the ones associated to the sites in the rectangle $Q_{2,1}(0)$, σ^3 the configuration with all the spins equal to minus one excepted the ones associated to the sites in $Q_{2,1}(0) \cup Q_1(0,1)$, and σ^4 the configuration with all the spins equal to minus one excepted the ones associated to the sites in the square $Q_2(0)$.

By definition, the path $\omega + \Omega_{\sigma^4}$ starts at $-\underline{1}$ and ends in $+\underline{1}$, i.e., $\omega + \Omega_{\sigma^4} \in \Theta(-\underline{1}, +\underline{1})$. Moreover, we shall prove that

$$\Phi_{\omega+\Omega_{\sigma^4}} - E(-\underline{1}) \leq \Gamma \tag{3.27}$$

The inequality (3.27), together with (3.26), implies (2.29).

We are then left with the proof of (3.27). We first consider the path ω; by using (2.18), see also Fig. 1, we get

$$\begin{aligned}
&E(-\underline{1}, \sigma^1) = E(-\underline{1}) + 2(5-h) && E(\sigma^1, -\underline{1}) = E(\sigma^1)\\
&E(\sigma^1, \sigma^2) = E(\sigma^1) + 2 \cdot 2(3-h) && E(\sigma^2, \sigma^1) = E(\sigma^2) + 2(1-h)\\
&E(\sigma^2, \sigma^3) = E(\sigma^2) + 2 \cdot 2(1-h) + 2(3-h) && E(\sigma^3, \sigma^2) = E(\sigma^3) + 2(1-h)\\
&E(\sigma^3, \sigma^4) = E(\sigma^3) + 3 \cdot 2(1-h) && E(\sigma^4, \sigma^3) = E(\sigma^4) + 2(1+h)
\end{aligned} \tag{3.28}$$

see Fig. 6 for a graphical representation.

By using (3.28), (2.26), (2.19), (2.12), and the definition (2.11) of the critical length λ, it is easy to show that, provided h is chosen smaller than $3 + \sqrt{5}$,

$$\Phi_\omega - E(-\underline{1}) \leq 34 - 14h < \Gamma \tag{3.29}$$

and

$$E(\sigma^4) - E(-\underline{1}) \leq 32 - 16h. \tag{3.30}$$

We consider, now, the path Ω_{σ^4}; by using (3.7) and (3.30), we get

$$\Phi_{\Omega_{\sigma^4}} - E(-\underline{1}) = \Phi_{\Omega_{\sigma^4}} - E(\sigma^4) + E(\sigma^4) - E(-\underline{1}) \leq \Gamma - 16(2-h) + 32 - 16h = \Gamma. \tag{3.31}$$

The inequality (3.27) follows from (3.29) and (3.31). This completes the proof of item 2 of Theorem 2.3. □

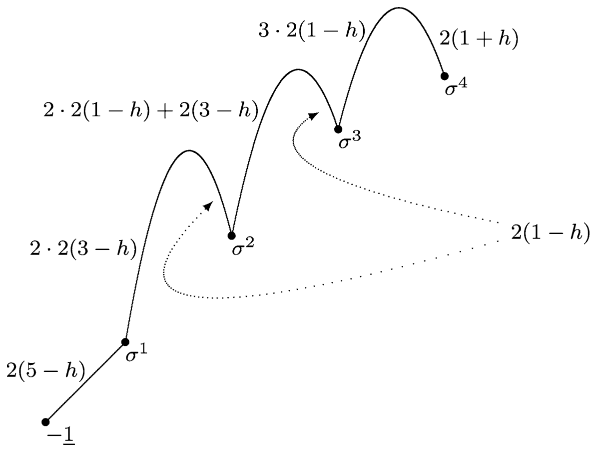

Fig. 6 Energy landscape for the path ω

Proof of item 3 of Theorem 2.3. The item follows from item 2 of Theorem 2.3 and item 3 of Proposition 3.2. □

4 Proof of Proposition 3.2

In Sect. 4.4 we shall prove Proposition 3.2 concerning the solution of the *minmax* problem. Some preliminary lemmata are stated in advance. More precisely, in Sect. 4.1 we state the Lemmata 4.1–4.4 concerning energy estimates for the maps A and B (see Sect. 3.2). In Sect. 4.2 the Lemmata 4.5 and 4.6, concerning properties of rectangular droplets (see Sect. 2.5), are stated. Sect. 4.3 is devoted to the comparison of configurations in $\mathcal{G}$ (see Sect. 3.2) and in $\mathcal{G}^{\mathrm{c}}$.

4.1 Energy Estimates for the Maps A and B

In Lemma 4.1 we give estimates on the energy of the configurations obtained by applying the maps A and B. For any $\sigma \in \mathcal{S}$ we let

$$N_A(\sigma) := \sum_{x\in\Lambda}[1-\delta_{\sigma(x),\overline{A}\sigma(x)}] \quad \text{and} \quad N_B(\sigma) := \sum_{x\in\Lambda}[1-\delta_{\overline{A}\sigma(x),\overline{B}\,\overline{A}\sigma(x)}] \tag{4.1}$$

with δ the Kronecker δ. Note that $N_A(\sigma)$ is the number of plus spins which are flipped by the iterative application of the map A to σ, while $N_B(\sigma)$ is the number of minus spins which are flipped by the iterative application of the bootstrap percolation map B to $\overline{A}\sigma$.

Lemma 4.1 *Let $\sigma \in \mathcal{S}$ and $h > 0$ small enough. Then*

1. *we have*

$$E(\sigma) \geq E(\overline{A}\sigma) + (2-10h)N_A(\sigma) \tag{4.2}$$

2. *we have*

$$E(\overline{A}\sigma) \geq E(\overline{B}\,\overline{A}\sigma) + 4hN_B(\sigma). \tag{4.3}$$

In order to prove Lemma 4.1 we state Lemma 4.2 on some properties of unstable plus spins and Lemma 4.3 concerning an energy estimate for a single application of the bootstrap percolation map B. Recall (3.1), (3.2), and (3.3); recall also that, given $\sigma \in \mathcal{S}$ and $x \in \Lambda$, the configuration σ^x has been defined in Sect. 2.2 as the one obtained by flipping the spin of σ associated with the site x.

Lemma 4.2 *Let $\sigma \in \mathcal{S}$; for $h > 0$ small enough, we have that the following statements hold true*:

1. *if there exists $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) > E(\sigma)$, then $|\partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)| \le 1$, i.e., there exists at most one nearest neighbor of x which is stable w.r.t. σ and such that the associated spin is minus one*;
2. *if there exists $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) \le E(\sigma)$, then $E(\sigma) \ge E(\sigma^x) + 2 - 10h$*;
3. *if $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_{\mathrm{u}}^+(\sigma)$, then*

$$2|\Lambda_{-1}^+(\sigma)| + 3|\Lambda_{-3}^+(\sigma)| \le 3|\Lambda_{+1}^-(\sigma)| + 4|\Lambda_{+3}^-(\sigma)|. \tag{4.4}$$

Proof of Lemma 4.2. Let $x \in \Lambda_{\mathrm{u}}^+(\sigma)$, then $\sigma(x) = +1$, $\sigma^x(x) = -1$, and $S_\sigma(x) < 0$; by using (2.9), we get

$$E(\sigma^x) - E(\sigma) = 2h - 2 + \sum_{y \in \partial\{x\}} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|). \tag{4.5}$$

Note that, since $\sigma(x) = +1$, we have that $S_\sigma(y)$, with $y \in \partial\{x\}$, can assume the values $-3, -1, +1, +3, +5$; by performing the direct computations one shows that

$$|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| \in \{-2, 2h, +2\} \tag{4.6}$$

for $y \in \partial\{x\}$.

Item 1. Let $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) > E(\sigma)$; since $S_\sigma(y) < 0$ for $y \in \partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)$, by using (4.5) we get

$$E(\sigma^x) - E(\sigma) = 2h - 2(1 + |\partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)|) + \sum_{y \in \partial\{x\} \setminus \Lambda_{\mathrm{s}}^-(\sigma)} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|).$$

Suppose, by the way of contradiction, that $|\partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)| \ge 2$, then we have

$$E(\sigma^x) - E(\sigma) \le 2h - 6 + \sum_{y \in \partial\{x\} \setminus \Lambda_{\mathrm{s}}^-(\sigma)} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|).$$

By (4.6) we obtain $|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| \le 2$ for $y \in \partial\{x\}$, and, noting that $|\partial\{x\} \setminus \Lambda_{\mathrm{s}}^-(\sigma)| \le 2$, we finally get $E(\sigma^x) - E(\sigma) \le 2h - 6 + 4 = 2h - 2 < 0$, which is in contradiction with the hypothesis.

Item 2. Let $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) \le E(\sigma)$. Recalling (4.5) and (4.6), we have that the number of sites $y \in \partial\{x\}$ such that $|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| = +2$ must be at most equal to the number of sites $y \in \partial\{x\}$ such that $|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| = -2$, otherwise it would be $E(\sigma^x) - E(\sigma) > 0$. Thus, un upper bound to the sum in (4.5) is found when all the $y \in \partial\{x\}$ are such that $|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| = 2h$. We then get

$$\sum_{y \in \partial\{x\}} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|) \le 2h|\partial\{x\}| = 8h$$

from which $E(\sigma^x) - E(\sigma) \le -2 + 10h$ follows.

Item 3. Consider $\sigma \in \mathcal{S}$ such that $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ and let $r_\sigma(y) = 1$ if $y \in \Lambda_{\mathrm{u}}^-$ and $r_\sigma(y) = 0$ otherwise. Recall that $\Lambda_{-5}^+(\sigma) = \emptyset$ and recall (3.3); by exploiting the first part of this lemma we get

$$\sum_{x\in\Lambda_{\mathrm{u}}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) = \sum_{x\in\Lambda_{-1}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) + \sum_{x\in\Lambda_{-3}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) \geq 2|\Lambda_{-1}^+(\sigma)| + 3|\Lambda_{-3}^+(\sigma)|.$$

On the other hand, a site in $\Lambda_{+1}^-(\sigma)$ is nearest neighbor of at most three sites in Λ_{u}^+, indeed the number of unstable pluses neighboring such a site can be less than three since some of the pluses can be stable ones, and a site in $\Lambda_{+3}^-(\sigma)$ is nearest neighbor of at most four sites in Λ_{u}^+; then we have

$$\sum_{x\in\Lambda_{\mathrm{u}}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) \leq 3|\Lambda_{+1}^-(\sigma)| + 4|\Lambda_{+3}^-(\sigma)|.$$

The inequality (4.4) follows trivially from the two bounds above. □

Lemma 4.3 *Suppose $h > 0$ small enough. Let $\sigma \in \mathcal{S}$, suppose $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_{\mathrm{u}}^+(\sigma)$. Then*

$$E(\sigma) \geq E(B\sigma) + 4h|\Lambda_{\geq -1}^-(\sigma)|. \tag{4.7}$$

Recall that $\Lambda_{\geq -1}^-(\sigma)$ is exactly the set of sites whose associated spin flips under the action of the bootstrap percolation map B (see (3.23)).

Proof of Lemma 4.3. To compare $E(\sigma)$ and $E(B\sigma)$ we shall use (2.19) and suitable bounds on $E(\sigma, B\sigma)$ and $E(B\sigma, \sigma)$. Recall (2.18), see also Fig. 1, and the definition (3.23) of the bootstrap percolation map B; we have that in the forward transition from σ to $B\sigma$ the energy costs are those associated to the flip of the stable minuses with two neighboring pluses and those associated to the permanence of the unstable pluses. More precisely, we have

$$E(\sigma, B\sigma) = E(\sigma) + 2(1-h)|\Lambda_{-1}^-(\sigma)| + 2(1-h)|\Lambda_{-1}^+(\sigma)| + 2(3-h)|\Lambda_{-3}^+(\sigma)|. \tag{4.8}$$

On the other hand, in the backward transition from $B\sigma$ to σ the energy costs that must be surely paid are those associated to the reverse flipping of the pluses that have been created in the forward transition; more precisely, we have

$$E(B\sigma, \sigma) \geq E(B\sigma) + 2(1+h)|\Lambda_{-1}^-(\sigma)| + 2(3+h)|\Lambda_{+1}^-(\sigma)| + 2(5+h)|\Lambda_{+3}^-(\sigma)|. \tag{4.9}$$

Note that in (4.9) it is not possible to take advantage from the permanence of the possible unstable pluses in $B\sigma$, because, as we shall see in the proof of item 2 of the Lemma 4.1, we have $\Lambda_{\mathrm{u}}^+(B\sigma) = \emptyset$.

To complete the proof we have to distinguish two cases. Suppose, first, that $\Lambda_{-1}^+(\sigma) = \Lambda_{+3}^-(\sigma) = \emptyset$; by using (4.8), (4.9), and (2.19), we get

$$E(\sigma) \geq E(B\sigma) + 4h|\Lambda_{-1}^-(\sigma)| - 2(3-h)|\Lambda_{-3}^+(\sigma)| + 2(3+h)|\Lambda_{+1}^-(\sigma)|.$$

The bound (4.7) follows noting that, in this case, $\Lambda_{\geq -1}^-(\sigma) = \Lambda_{-1}^-(\sigma) \cup \Lambda_{+1}^-(\sigma)$ and (4.4) reduces to $|\Lambda_{-3}^+(\sigma)| \leq |\Lambda_{+1}^-(\sigma)|$. Suppose, now, that either $\Lambda_{-1}^+(\sigma) \neq \emptyset$ or $\Lambda_{+3}^-(\sigma) \neq \emptyset$. By using (4.4) we have

$$
\begin{aligned}
|\Lambda^+_{-1}(\sigma)| + 3|\Lambda^+_{-3}(\sigma)| &\le 2|\Lambda^+_{-1}(\sigma)| + 3|\Lambda^+_{-3}(\sigma)| \\
&\le 3|\Lambda^-_{+1}(\sigma)| + 4|\Lambda^-_{+3}(\sigma)| \le 3|\Lambda^-_{+1}(\sigma)| + 5|\Lambda^-_{+3}(\sigma)|.
\end{aligned}
$$

Since either $|\Lambda^+_{-1}(\sigma)| \ge 0$ or $|\Lambda^-_{+3}(\sigma)| \ge 0$, we have that

$$
|\Lambda^+_{-1}(\sigma)| + 3|\Lambda^+_{-3}(\sigma)| < 3|\Lambda^-_{+1}(\sigma)| + 5|\Lambda^-_{+3}(\sigma)|. \tag{4.10}
$$

We shall prove that, provided $h < 1$,

$$
(1-h)|\Lambda^+_{-1}(\sigma)| + (3-h)|\Lambda^+_{-3}(\sigma)| < (3-h)|\Lambda^-_{+1}(\sigma)| + (5-h)|\Lambda^-_{+3}(\sigma)|. \tag{4.11}
$$

First of all we note that the inequality (4.11) is equivalent to

$$
\begin{aligned}
&h[|\Lambda^-_{+3}(\sigma)| + |\Lambda^-_{+1}(\sigma)| - |\Lambda^+_{-1}(\sigma)| - |\Lambda^+_{-3}(\sigma)|] \\
&\quad < |\Lambda^-_{+3}(\sigma)| + |\Lambda^-_{+1}(\sigma)| - |\Lambda^+_{-1}(\sigma)| - |\Lambda^+_{-3}(\sigma)| + [2|\Lambda^-_{+1}(\sigma)| + 4|\Lambda^-_{+3}(\sigma)| \\
&\qquad - 2|\Lambda^+_{-3}(\sigma)|]
\end{aligned}
$$

which is trivially satisfied when the left hand side is negative or equal to zero, since (4.10) implies that the right hand side is strictly positive; on the other hand, if the left hand side is strictly positive, recalling that $h < 1$, the inequality will follow once we shall have proved that $2|\Lambda^-_{+1}(\sigma)| + 4|\Lambda^-_{+3}(\sigma)| - 2|\Lambda^+_{-3}(\sigma)| \ge 0$.

To get this last bound we note that by using item 1 in Lemma 4.2, it follows that for each site belonging to Λ^+_{-3}, there are at least three unstable minus spins among the four nearest neighboring ones. Hence, we get $|\Lambda^-_{\mathrm{u}}(\sigma)| \ge (4/3)|\Lambda^+_{-3}(\sigma)|$. Moreover, noted that $|\Lambda^-_{\mathrm{u}}(\sigma)| = |\Lambda^-_{+1}(\sigma)| + |\Lambda^-_{+3}(\sigma)|$, we also get

$$
\begin{aligned}
&2|\Lambda^-_{+1}(\sigma)| + 4|\Lambda^-_{+3}(\sigma)| - 2|\Lambda^+_{-3}(\sigma)| \\
&\quad = 2|\Lambda^-_{+3}(\sigma)| + 2\big[|\Lambda^-_{+1}(\sigma)| + |\Lambda^-_{+3}(\sigma)| - |\Lambda^+_{-3}(\sigma)|\big] \\
&\quad = 2|\Lambda^-_{+3}(\sigma)| + 2\big[|\Lambda^-_{u}(\sigma)| - |\Lambda^+_{-3}(\sigma)|\big] \ge 2\big[|\Lambda^-_{+3}(\sigma)| + 1/3|\Lambda^+_{-3}(\sigma)|\big] \ge 0.
\end{aligned}
$$

Finally, the bound (4.7) follows easily by using (2.19), (4.8), (4.9), and the inequality (4.11). □

Proof of Lemma 4.1. Item 1. The bound (4.2) is proven easily by applying iteratively item 2 of Lemma 4.2.

Item 2. Suppose $\overline{B}\,\overline{A}\sigma = B^n\overline{A}\sigma$ for some integer n. We first note that by Lemma 4.2 each site $x \in \Lambda^+_{\mathrm{u}}(\overline{A}\sigma)$ has at least two neighboring minuses which are unstable w.r.t. $\overline{A}\sigma$, more precisely $|\partial\{x\} \cap \Lambda^-_{\mathrm{u}}(\overline{A}\sigma)| \ge 2$. Recall the definition (3.23) of the bootstrap percolation map B; since $\Lambda^-_{\mathrm{u}}(\overline{A}\sigma) \subset \Lambda^-_{\ge -1}(\overline{A}\sigma)$, the minuses in $\partial\{x\} \cap \Lambda^-_{\mathrm{u}}(\overline{A}\sigma)$ flip under the action of B. Hence, $|\partial\{x\} \cap \Lambda^+(B\overline{A}\sigma)| \ge 2$. We then have $\Lambda^+_{\mathrm{u}}(B\overline{A}\sigma) = \emptyset$; in other words all the unstable pluses in $\overline{A}\sigma$ become stable after the application of a single step of the bootstrap percolation.

By definition of the bootstrap percolation map we also have that $\Lambda^+_{\mathrm{u}}(B^i\overline{A}\sigma) = \emptyset$ for any $i = 2, \dots, n$, i.e., no site in $\Lambda^+(B^i\overline{A}\sigma)$ is unstable w.r.t. $B^i\overline{A}\sigma$. Note, finally, that $E((\overline{A}\sigma)^x) > E(\overline{A}\sigma)$ for any $x \in \Lambda^+_{\mathrm{u}}(\overline{A}\sigma)$. The theorem then follows by applying iteratively Lemma 4.3. □

Let $\sigma \in \mathcal{S}$, we refine the estimate (4.2) by considering the plus spins that are flipped by the iterative application of the map A and are associated with sites outside the support of the configuration $\overline{B}\,\overline{A}\sigma$. Let the *branch* of σ be

$$L(\sigma) := |\Lambda^+(\sigma) \setminus \Lambda^+(\overline{B}\,\overline{A}\sigma)| \tag{4.12}$$

i.e., the number of pluses outside the rectangles of $\overline{B}\,\overline{A}\sigma$ which are flipped by the map A; note that $L(\sigma) \le N_A(\sigma)$ (see (4.1)).

Lemma 4.4 *For any $\sigma \in \mathcal{S}$ such that $L(\sigma) \ge 1$, we have that*

$$E(\sigma) - E(\overline{A}\sigma) \ge \begin{cases} 6-2h & \text{if } L(\sigma)=1 \\ 10-6h+(2-10h)(L(\sigma)-2) & \text{if } L(\sigma) \ge 2. \end{cases} \tag{4.13}$$

Proof of Lemma 4.4. Let $\sigma \in \mathcal{S}$ such that $L(\sigma)=1$, the set $\Lambda^+(\sigma) \setminus \Lambda^+(\overline{B}\,\overline{A}\sigma)$ has a unique element x. There exists a natural number j such that $A^{j-1}\sigma(x)=+1$ and $A^j\sigma(x)=-1$. For $y \in \partial\{x\} \cap (\Lambda^+(\overline{B}\,\overline{A}\sigma))^{\mathrm{c}}$ we have $|S_{A^j\sigma}(y)+h| - |S_{A^{j-1}\sigma}(y)+h| = 2$, while for $y \in \partial\{x\} \cap \Lambda^+(\overline{B}\,\overline{A}\sigma)$ we have the trivial bound $|S_{A^j\sigma}(y)+h| - |S_{A^{j-1}\sigma}(y)+h| \ge -2$. Since $|\partial\{x\} \cap (\Lambda^+(\overline{B}\,\overline{A}\sigma))^{\mathrm{c}}| \ge 3$ and $|\partial\{x\} \cap \Lambda^+(\overline{B}\,\overline{A}\sigma)| \le 1$, by using (2.9) we get

$$\begin{aligned} E(A^{j-1}\sigma) - E(A^j\sigma) = 2-2h &+ \sum_{y \in \partial\{x\} \cap (\Lambda^+(\overline{B}\,\overline{A}\sigma))^{\mathrm{c}}} (|S_{A^j\sigma}(y)+h| - |S_{A^{j-1}\sigma}(y)+h|) \\ &+ \sum_{y \in \partial\{x\} \cap \Lambda^+(\overline{B}\,\overline{A}\sigma)} (|S_{A^j\sigma}(y)+h| - |S_{A^{j-1}\sigma}(y)+h|) \ge 6-2h. \end{aligned} \tag{4.14}$$

Recall, finally, that by definition the map A decreases the energy; then, by (4.14), we have

$$E(\sigma) \ge E(A^{j-1}\sigma) \ge E(A^j\sigma) - 2h + 6 \ge E(\overline{A}\sigma) - 2h + 6$$

and the bound (4.13) follows.

Let now $\sigma \in \mathcal{S}$ such that $L(\sigma)=2$; the set $\Lambda^+(\sigma) \setminus \Lambda^+(\overline{B}\,\overline{A}\sigma)$ has two elements x, y. Since $\overline{B}\,\overline{A}\sigma = \overline{B}\,\overline{A}\sigma^y$ and $L(\sigma)=2$, we have $L(\sigma^y)=1$; by using $\overline{A}\sigma^y = \overline{A}\sigma$ and (4.13) in the already proven case we have that

$$E(\sigma) - E(\overline{A}\sigma) = E(\sigma) - E(\sigma^y) + E(\sigma^y) - E(\overline{A}\sigma) \ge E(\sigma) - E(\sigma^y) + 6 - 2h. \tag{4.15}$$

In order to bound $E(\sigma) - E(\sigma^y)$, we first note that by (2.9) we get

$$E(\sigma) - E(\sigma^y) = -2h - \sum_{z \in \overline{\{y\}}} (|S_\sigma(z)+h| - |S_{\sigma^y}(z)+h|). \tag{4.16}$$

We distinguish, now, two cases. We first suppose that $x \notin \overline{\{y\}}$, i.e., the two sites x and y are not nearest neighbors. It is easy to prove that $-(|S_\sigma(y)+h| - |S_{\sigma^y}(y)+h|) = +2$. Moreover, note that the contribution to the sum (4.16) of all the sites in $\partial\{y\} \cap (\Lambda^+(\overline{B}\,\overline{A}\sigma))^{\mathrm{c}}$ is equal to $+2$ excepted for at most one site whose contribution is equal to $-2h$. Note, also, that $|\partial\{y\} \cap (\Lambda^+(\overline{B}\,\overline{A}\sigma))^{\mathrm{c}}| \ge 3$; hence, we have that $E(\sigma) - E(\sigma^y) \ge -2h + (2-2h) + 2 + 2 - 2h - 2$, where the contribution of the site $\partial\{y\} \cap \Lambda^+(\overline{B}\,\overline{A}\sigma)$, which possibly exists, has been bounded trivially by -2. The bound (4.13) follows immediately.

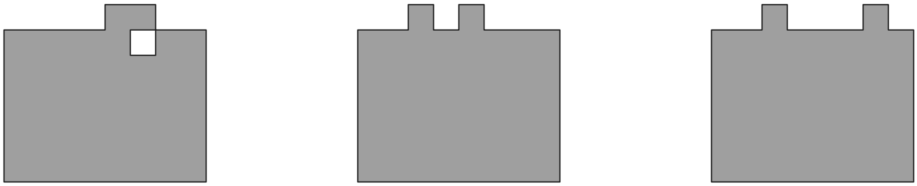

Fig. 7 The three cases studied in the proof of the Lemma 4.4; *on the left* the not trivial one

Suppose, now, that $x \in \overline{\{y\}}$, i.e., the two sites x and y are adjacent. The only not trivial case, see Fig. 7, is the one in which both the sites x and y are at distance one from the set $\Lambda^+(\overline{B}\,\overline{A}\sigma)$. Since the plus spins associated to x and y are flipped by the iterative application of the map A to σ, the spin associated to at least one of the two sites in $\partial\{x, y\} \cap \Lambda^+(\overline{B}\,\overline{A}\sigma)$ is equal to -1, see Fig. 7. Without loss of generality we let $\partial\{y\} \cap \Lambda^+(\overline{B}\,\overline{A}\sigma) = \{y'\}$ and $\sigma(y') = -1$. It is easy to prove that $-(|S_\sigma(y) + h| - |S_{\sigma^y}(y) + h|) = +2$, $-(|S_\sigma(x) + h| - |S_{\sigma^y}(x) + h|) \geq -2h$, $-(|S_\sigma(y') + h| - |S_{\sigma^y}(y') + h|) \geq -2$, and $-(|S_\sigma(z) + h| - |S_{\sigma^y}(z) + h|) = 2$ for each $z \in \partial\{y\} \setminus \{x, y'\}$. Hence, by using (4.16) we get

$$E(\sigma) - E(\sigma^y) \geq -2h + 2 - 2h - 2 + 2 + 2 = 4 - 4h. \tag{4.17}$$

The bound (4.13) follows by (4.17) and (4.15).

Let, finally, $\sigma \in \mathcal{S}$ such that $L(\sigma) \geq 3$. Let i a suitable integer such that $L(A^i\sigma) = 2$. The bound (4.13) follows easily by using the Lemma 4.1 and (4.13) applied to $A^i\sigma$. □

4.2 Energy Estimates for Rectangular Droplets

We first state and prove the following Lemma on some simple geometrical properties of rectangles on the lattice.

Lemma 4.5 *Let Q_{l_i,m_i}, for $i = 1, \dots, n$, be pairwise disjoint rectangles with sides $l_i, m_i \in \mathbb{N} \setminus \{0\}$, such that $\ell_i \leq m_i$ for $i = 1, \dots, n$, and semi-perimeter $p := \sum_i^n (\ell_i + m_i)$.*

1. *We have*

$$\frac{1}{4}p^2 \geq \sum_{i=1}^{n} l_i\, m_i. \tag{4.18}$$

2. *If there exists a positive integer k such that $\ell_i \leq k - 1$ and $m_i \leq k$ for all $i = 1, \dots, n$, we have*

$$\sum_{i=1}^{n} \ell_i m_i \leq \frac{1}{2}kp - \frac{1}{2}\sum_{i=1}^{n} m_i. \tag{4.19}$$

3. *If $n \geq 2$ and $l_i \geq 2$ then*

$$\frac{1}{4}p^2 \geq \sum_{i=1}^{n} l_i\, m_i + p. \tag{4.20}$$

Proof of Lemma 4.5. Item 1: we have

$$\frac{1}{4}p^2 = \frac{1}{4}\left(\sum_{i=1}^{n}(l_i + m_i)\right)^2 \geq \frac{1}{4}\sum_{i=1}^{n}(l_i + m_i)^2 = \frac{1}{4}\sum_{i=1}^{n}(l_i - m_i)^2 + \sum_{i=1}^{n} l_i\, m_i \geq \sum_{i=1}^{n} l_i\, m_i.$$

Item 2: we have

$$\sum_{i=1}^{n}\ell_i m_i = 2\sum_{i=1}^{n}\frac{1}{2}l_i m_i \le \frac{1}{2}\sum_{i=1}^{n}(k-1)m_i + \frac{1}{2}\sum_{i=1}^{n}\ell_i k \le \frac{1}{2}k\sum_{i=1}^{n}(\ell_i+m_i) - \frac{1}{2}\sum_{i=1}^{n}m_i$$

which implies (4.19). Item 3: note that

$$\begin{aligned}\left(\frac{1}{2}\sum_{i=1}^{n}(l_i+m_i)\right)^2 - \sum_{i}^{n} l_i\, m_i &= \frac{1}{4}\left(\left(\sum_{i=1}^{n}(l_i+m_i)\right)^2 - 4\sum_{i}^{n} l_i\, m_i\right)\\ &= \frac{1}{4}\left(\sum_{i}^{n}(l_i+m_i)^2 - 4\sum_{i=1}^{n} l_i\, m_i + \sum_{i\neq j}(l_i+m_i)(l_j+m_j)\right)\\ &= \frac{1}{4}\left(\sum_{i=1}^{n}(l_i-m_i)^2 + \sum_{i\neq j}(l_i+m_i)(l_j+m_j)\right)\\ &\ge \frac{4}{4}\sum_{j=1}^{n}(l_j+m_j) = p\end{aligned}$$

where in the second step we used $n \ge 2$ and in the last step $l_i \wedge m_i \ge 2$. The bound (4.20) follows. □

We introduce the notion of semi-perimeter of a multi-rectangular droplet. Let $n \ge 1$ and $\ell_1, m_1, \dots, \ell_n, m_n$ integers such that $2 \le \ell_1, m_1, \dots, \ell_n, m_n \le L-2$, $\sigma \in \mathcal{S}$ a n-rectangular droplet with sides $\ell_1, m_1, \dots, \ell_n, m_n$, we let

$$p(\sigma) := \sum_{i=1}^{n}(\ell_i + m_i) \tag{4.21}$$

be the *semi-perimeter* of the multi-rectangular droplet σ.

Lemma 4.6 *Let ℓ, m two integers such that $2 \le \ell \le m \le L-2$ and $\sigma \in \mathcal{S}$ a rectangular droplet with sides ℓ and m. If $\ell \le \lambda - 1$, we have*

$$E(\sigma) - E(-\underline{1}) > 8\ell > 0. \tag{4.22}$$

If $\ell \le \lambda - 1$ and $m \ge \lambda + 1$, we have

$$E(\sigma) - E(\mathcal{R}) \ge 4h(1-\delta_h) > 0 \tag{4.23}$$

where we recall $\mathcal{R}$ has been defined above Proposition 3.2 and δ_h below (2.11).

Moreover, for $n \ge 1$ integer, for any n-rectangular droplet $\eta \in \mathcal{S}$ with sides $2 \le \ell_i \le m_i$ such that $\ell_i \le \lambda - 1$ and $m_i \le \lambda$ for $i = 1, \dots, n$, we have that

$$E(\eta) - E(-\underline{1}) > (4-2h)p(\eta) + \frac{1}{2}\sum_{i=1}^{n}m_i. \tag{4.24}$$

Proof of Lemma 4.6. Suppose $\ell \le \lambda - 1$: by using (2.25) we have $E(\sigma) - E(-\underline{1}) = -4h\ell m + 8(\ell+m) = (8-4h\ell) + 8\ell$; since $\ell \le \lambda - 1$, the lemma follows. Suppose $\ell \le \lambda - 1$

and $m \geq \lambda + 1$, by using (2.25) we have $E(\sigma) - E(\mathcal{R}) = 4h(m-\lambda)[(\lambda-\ell) - \delta_h]$, which implies (4.23).

We finally prove (4.24). Recall that by hypothesis $\ell_i \leq \lambda - 1$ and $m_i \leq \lambda$ for any $i = 1, \dots, n$; by definition of multi-rectangular droplets and by using (4.19) with $k = \lambda$, we have

$$|\Lambda^+(\eta)| \leq \frac{\lambda}{2} p(\eta) - \frac{1}{2} \sum_{i=1}^{n} m_i. \tag{4.25}$$

Now, by using (2.25), (4.21), (4.19), and the fact that the support of a multi-rectangular droplet is made of pairwise not interacting rectangles, we have that

$$E(\eta) - E(-\underline{1}) = -4h|\Lambda^+(\eta)| + 8p(\eta) \geq p(\eta)(8 - 2h\lambda) + \frac{1}{2} \sum_{i=1}^{n} m_i$$

which implies (4.24) since $\lambda < (2/h) + 1$. □

4.3 Relations between Configurations in $\mathcal{G}$ and in $\mathcal{G}^c$

Consider $\sigma \in \mathcal{G}$ and $\eta \in \mathcal{G}^c$, in Lemma 4.7 we state a property relating the pluses in η to those in $\overline{B}\,\overline{A}\sigma$ and we bound from below the transition rate $\Delta(\sigma, \eta)$ (see(2.20)).

Lemma 4.7 *Let $\sigma \in \mathcal{G}$ and $\eta \notin \mathcal{G}$,*

1. *We have*

$$|\Lambda^+(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}(\sigma))| \geq 2. \tag{4.26}$$

2. *We have*

$$\Delta(\sigma, \eta) \geq \begin{cases} 12 - 4h & \text{for } L(\sigma) = 0 \\ 4 - 4h & \text{for } L(\sigma) = 1. \end{cases} \tag{4.27}$$

Proof of Lemma 4.7. Item 1: the item follows from the definition of the subcritical set $\mathcal{G}$. Indeed, if $|\Lambda^+(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}(\sigma))| \leq 1$, we have that under the map A the positive spin outside $\Lambda^+(\overline{B}\,\overline{A}\sigma)$ is flipped, so that $\Lambda^+(\overline{B}\,\overline{A}\eta) \subseteq \Lambda^+(\overline{B}\,\overline{A}\sigma)$. Hence $\eta \in \mathcal{G}$, that is a contradiction.

Item 2: from (2.20) we get

$$\Delta(\sigma, \eta) = 2 \sum_{z \in \Lambda:\, \eta(z)\, S_\sigma(z) < 0} |S_\sigma(z) + h| \geq 2 \sum_{z \in \Lambda \setminus \Lambda^+(\overline{B}\,\overline{A}\sigma):\, \eta(z)\, S_\sigma(z) < 0} |S_\sigma(z) + h|. \tag{4.28}$$

If $L(\sigma) = 0$, by (4.26),(4.28), the theorem follows. Indeed, in the r.h.s of (4.28) there are at least two terms corresponding to sites x and y such that $\eta(x) = \eta(y) = 1$, and $S_\sigma(x) \leq -3$, $S_\sigma(y) \leq -3$. If $L(\sigma) = 1$, from (4.26) there exist two sites

$$\{x, y\} \subseteq \Lambda^+(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}\sigma). \tag{4.29}$$

Note that, since $L(\sigma) = 1$, one has $S_\sigma(x) \leq -1$ and $S_\sigma(y) \leq -1$. From (4.28) we have the bound

$$\Delta(\sigma, \eta) \geq 2(1-h) + 2(1-h) \tag{4.30}$$

and the theorem follows, see also Fig. 1. □

4.4 Proof of the Proposition 3.2

Let $\sigma \in \mathcal{S}$ and suppose $\overline{B}\,\overline{A}\sigma \neq -\underline{1}$, there exist $n(\sigma) \in \mathbb{N} \setminus \{0\}$, $\ell_i(\sigma), m_i(\sigma)$ integers larger than 2, and $x_i(\sigma) \in \Lambda$ for $i = 1, \ldots, n(\sigma)$ such that

$$\Lambda^+(\overline{B}\,\overline{A}\sigma) = \bigcup_{i=1}^{n(\sigma)} Q_{\ell_i(\sigma), m_i(\sigma)}(x_i(\sigma)).$$

If $\overline{B}\,\overline{A}\sigma = -\underline{1}$ we shall understand $n(\sigma) = 1$, $\ell_1(\sigma) = m_1(\sigma) = 0$, and $p(\sigma) = 0$, see also (4.21). Let $\sigma \in \mathcal{S}$, we order the droplets in $\Lambda^+(\overline{B}\,\overline{A}\sigma)$ so that $\ell_i(\sigma) \wedge m_i(\sigma) \geq \lambda$ for $i = 1, \ldots, k(\sigma)$ and $\ell_i(\sigma) \wedge m_i(\sigma) \leq \lambda - 1$ for $i = k(\sigma) + 1, \ldots, n(\sigma)$; note that for $\sigma \in \mathcal{G}$ we have $k(\sigma) = 0$, while for $\sigma \in \mathcal{G}^c$ we have $k(\sigma) \geq 1$. For the sake of simplicity, for $\sigma \in \mathcal{G}^c$ in the sequel we shall let $r_i(\sigma) := \ell_i(\sigma) - \lambda$ and $q_i(\sigma) := m_i(\sigma) - \lambda$ for $i = 1, \ldots, k(\sigma)$.

Before starting the proof of the Proposition 3.2 we sketch the main idea. We shall define the subsets of the configuration space $\mathcal{A}_5 \subset \mathcal{A}_4 \subset \mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{G}$, $\mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{G}^c$, and reduce the proof to the computation of $E(\eta, \zeta)$ for $\eta \in \mathcal{A}_5$ and $\zeta \in \mathcal{B}_2$ (see Fig. 8). We recall (4.21), (4.12), (4.1), and let

$$\begin{aligned}
\mathcal{A}_1 &:= \{\sigma \in \mathcal{G} : \ell_i(\sigma) \vee m_i(\sigma) \leq \lambda \text{ for } i = 1, \ldots, n(\sigma)\} \\
\mathcal{A}_2 &:= \{\sigma \in \mathcal{A}_1 : p(\sigma) \leq 2\lambda + 4,\ L(\sigma) \leq 4\lambda + 42\} \\
\mathcal{A}_3 &:= \{\sigma \in \mathcal{A}_2 : p(\sigma) \geq 2\lambda - 50\} \\
\mathcal{A}_4 &:= \{\sigma \in \mathcal{A}_3 : n(\sigma) = 1\} \\
\mathcal{A}_5 &:= \{\sigma \in \mathcal{A}_4 : p(\sigma) = 2\lambda - 1\}
\end{aligned} \tag{4.31}$$

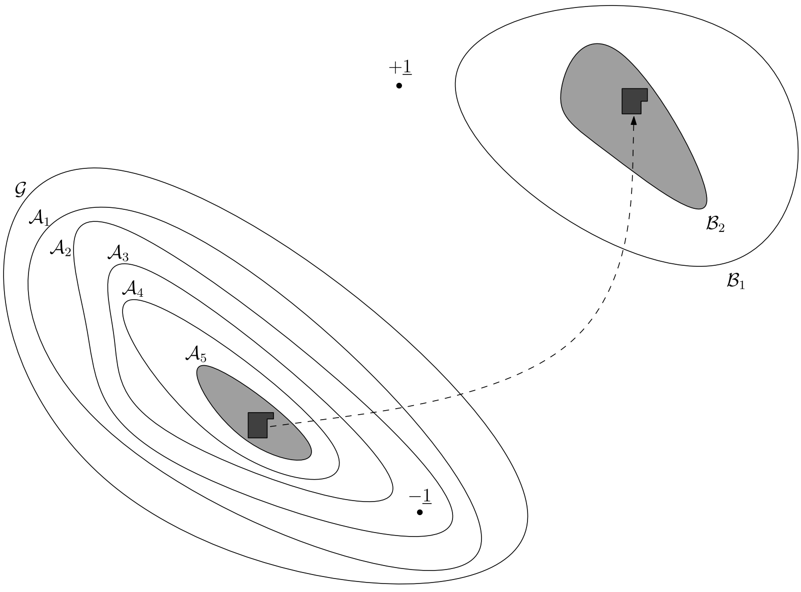

Fig. 8 Restricted sets on which we evaluate $E(\eta, \zeta)$ in the proof of item 2 of Proposition 3.2

and

$$\begin{aligned} \mathcal{B}_1 &:= \{\sigma \in \mathcal{G}^c : \ell_i(\sigma), m_i(\sigma) \le L-2 \text{ for } i=1,\dots,n(\sigma)\} \\ \mathcal{B}_2 &:= \{\sigma \in \mathcal{B}_1 : 4hN_B(\sigma) - 4h\sum_{i=1}^{k(\sigma)} (r_i(\sigma)+q_i(\sigma)+r_i(\sigma)q_i(\sigma)) \le 10-2h\}. \end{aligned} \tag{4.32}$$

In order to bound $E(\eta,\zeta)$ for $\eta \in \mathcal{G}$ and $\zeta \in \mathcal{G}^c$, we shall use the identity

$$\begin{aligned} E(\eta,\zeta) - E(-\underline{1}) = [E(\eta) - E(\overline{A}\eta)] + [E(\overline{A}\eta) - E(\overline{B}\,\overline{A}\eta)] \\ + [E(\overline{B}\,\overline{A}\eta) - E(-\underline{1})] + \Delta(\eta,\zeta) \end{aligned} \tag{4.33}$$

which is a straightforward consequence of the definition (2.20). Depending on the choice of η, the different terms in the r.h.s. of the identity (4.33) will be properly bounded in order to get the theorem.

Proof of Proposition 3.2. Item 1. The proof is an immediate application of the definition of the set $\mathcal{G}$ (see Sect. 3.2).

Items 2. *Step 1.* Let $\eta \in \mathcal{G} \setminus \mathcal{A}_1$ and $\zeta \in \mathcal{G}^c$. There exists $i \in \{1,\dots,n(\eta)\}$ such that $l_i(\eta) \vee m_i(\eta) \ge \lambda+1$; hence, by using (4.33), (4.3), $N_B(\eta) \ge 0$, (4.22), and (4.23), we get

$$E(\eta,\zeta) - E(-\underline{1}) \ge [E(\eta) - E(\overline{A}\eta)] + [E(\mathcal{R}) - E(-\underline{1})] + \Delta(\eta,\zeta) \tag{4.34}$$

where (4.22) and (4.23) have been applied to each non-interacting droplet in $\overline{B}\,\overline{A}\sigma$ to deduce that $E(\overline{B}\,\overline{A}\sigma) - E(\mathcal{R}) \ge 0$. Now, if $L(\eta)=0$, by using (4.34), (4.2), $N_A(\eta) \ge 0$, (4.27), and (3.24), we get

$$E(\eta,\zeta) - E(-\underline{1}) > [E(\mathcal{R}) - E(-\underline{1})] + 12 - 4h > \Gamma.$$

On the other hand, if $L(\eta) \ge 1$, by using (4.34), (4.13), (4.27), and (3.24), we get

$$E(\eta,\zeta) - E(-\underline{1}) > 6 - 2h + [E(\mathcal{R}) - E(-\underline{1})] + 4 - 4h > \Gamma.$$

Step 2. Let $\eta \in \mathcal{A}_1 \setminus \mathcal{A}_2$ and $\zeta \in \mathcal{G}^c$. By using (4.33), (4.3), and $N_B(\eta) \ge 0$, we get

$$E(\eta,\zeta) - E(-\underline{1}) \ge [E(\eta) - E(\overline{A}\eta)] + [E(\overline{B}\,\overline{A}\eta) - E(-\underline{1})] + \Delta(\eta,\zeta). \tag{4.35}$$

Now, suppose $p(\eta) \ge 2\lambda+5$, by using (4.35), (4.24), $\Delta(\eta,\zeta) \ge 0$, the definition (2.11), and (2.13), we get

$$E(\eta,\zeta) - E(-\underline{1}) > (4-2h)(2\lambda+5) > 8\lambda + 12 - 14h > \Gamma$$

provided $h>0$ is chosen smaller than $1/6$. Suppose, finally, $L(\eta) \ge 4\lambda+43$. If $\overline{B}\,\overline{A}\eta \ne -\underline{1}$, by using (4.22) we get $E(\overline{B}\,\overline{A}\eta) - E(-\underline{1}) \ge 0$; note that this bound holds trivially also in the case $\overline{B}\,\overline{A}\eta = -\underline{1}$. Hence, by using this bound, (4.35), (4.13), and (2.13), we get

$$E(\eta,\zeta) - E(-\underline{1}) > 10 - 6h + (2-10h)(4\lambda+43) > \Gamma.$$

Step 3. Let $\eta \in \mathcal{A}_2$ and $\zeta \in \mathcal{G}^c \setminus \mathcal{B}_1$. There exists $i \in \{1,\dots,k(\zeta)\}$ such that $\ell_i(\zeta) \vee m_i(\zeta) > L-2$. Since $\eta \in \mathcal{A}_2$ we have that $p(\eta) \le 2\lambda+4$ and $L(\eta) \le 4\lambda+42$, then by using

(4.19) with $k=\lambda$ we have $|\Lambda^+(\eta)|\le|\Lambda^+(\overline{B}\,\overline{A}\eta)|+L(\eta)\le\lambda p(\eta)/2+L(\eta)\le\lambda^2+6\lambda+42$. Given the magnetic field $h>0$, the number of plus spins in η is bounded by a finite number; then we can choose $L=L(h)$ so large that there exist an horizontal and a vertical stripe winding around the torus with arbitrarily large width and such that $\eta(x)$ is equal to -1 for each x in such two stripes. Since in $\overline{B}\,\overline{A}\zeta$, there exists a rectangular droplet of pluses with one of the two side lengths larger or equal to $L-2$; we choose L so large that $\Delta(\eta,\zeta)>\Gamma$. By using, finally, (2.20) we get $E(\eta,\zeta)-E(-\underline{1})>\Gamma$, once we remark that $E(\eta)-E(-\underline{1})\ge E(\overline{B}\,\overline{A}\eta)-E(-\underline{1})\ge 0$.

Step 4. Let $\eta\in\mathcal{A}_2$ and $\zeta\in\mathcal{B}_1\setminus\mathcal{B}_2$. By using Lemma 4.1 and $N_A(\zeta)\ge 0$, we have the bound

$$E(\zeta)-E(-\underline{1})\ge E(\overline{B}\,\overline{A}\zeta)-E(-\underline{1})+4hN_B(\zeta). \tag{4.36}$$

By (2.25) and (2.12) it follows

$$\begin{aligned}
E(\overline{B}\,\overline{A}\zeta)-E(-\underline{1}) &= -4h\sum_{i=1}^{n(\zeta)}(\lambda+r_i(\zeta))(\lambda+q_i(\zeta))+8\sum_{i=1}^{n(\zeta)}(2\lambda+r_i(\zeta)+q_i(\zeta))\\
&= n(\zeta)(\Gamma-10+6h)-\sum_{i=1}^{n(\zeta)}(r_i(\zeta)+q_i(\zeta))(4h\lambda-8)\\
&\quad -4n(\zeta)(h\lambda-2)-4h\sum_{i=1}^{n(\zeta)}r_i(\zeta)q_i(\zeta)\\
&> (\Gamma-10+2h)-4h\sum_{i=1}^{n(\zeta)}(r_i(\zeta)+q_i(\zeta)+r_i(\zeta)q_i(\zeta))
\end{aligned} \tag{4.37}$$

where in the last inequality we used (2.11) and the fact that $\Gamma>10-6h$. Hence, by (4.36) and (4.37), we have

$$E(\zeta)-E(-\underline{1})\ge\Gamma-(10-2h)+4hN_B(\zeta)-4h\sum_{i=1}^{n(\zeta)}(r_i(\zeta)+q_i(\zeta)+r_i(\zeta)q_i(\zeta)).$$

Since $\zeta\in\mathcal{B}_2\setminus\mathcal{B}_1$, we get $E(\zeta)-E(-\underline{1})>\Gamma$. Finally, by the inequality in (2.19), we get $E(\eta,\zeta)-E(-\underline{1})>\Gamma$.

Step 5. Let $\eta\in\mathcal{A}_2\setminus\mathcal{A}_3$ and $\zeta\in\mathcal{B}_2$. We note now that $E(\overline{B}\,\overline{A}\eta)-E(-\underline{1})\ge 0$, which is trivial if $\overline{B}\,\overline{A}\eta=-\underline{1}$, otherwise it follows immediately from (4.22). By using this bound, (4.33), Lemma 4.1, $N_A(\eta)\ge 0$, and $N_B(\eta)\ge 0$, we get

$$E(\eta,\zeta)-E(-\underline{1})\ge\Delta(\eta,\zeta). \tag{4.38}$$

We find, now, a lower bound to $\Delta(\eta,\zeta)$ by multiplying the minimum *quantum* $2(1-h)$, see Fig. 1, times the number of flips against the drift in the transition from η to ζ. More precisely,

$$\begin{aligned}
\Delta(\eta,\zeta) &\ge 2(1-h)\,|\{x\in\Lambda:\,\eta(x)S_\eta(x)>0,\ \eta(x)\zeta(x)<0\}|\\
&\ge 2(1-h)(|\Lambda^+(\zeta)|-|\bar{\Lambda}(\eta)|)
\end{aligned} \tag{4.39}$$

with

$$\bar{\Lambda}(\eta) := \Lambda^+(\overline{B}\,\overline{A}\eta) \cup \overline{\Lambda^+(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)} \tag{4.40}$$

where we recall the closure of a subset of the lattice has been defined in Sect. 2.1.

Recalling that the application of the map A does not add pluses, the number of plus spins in the configuration ζ can be bounded from below by adding the number of pluses in $\overline{B}\,\overline{A}\zeta$ to the branch $L(\zeta)$ of ζ and subtracting the number of pluses $N_B(\zeta)$ added by the bootstrap map B. Namely, we have

$$|\Lambda^+(\zeta)| \ge n(\zeta)\lambda^2 + \lambda \sum_{i=1}^{n(\zeta)} (r_i(\zeta) + q_i(\zeta)) + \sum_{i=1}^{n(\zeta)} r_i(\zeta) q_i(\zeta) - N_B(\zeta) + L(\zeta).$$

Now, by using that $\zeta \in \mathcal{B}_2$, (2.11), and $L(\zeta) \ge 0$, we get

$$\begin{aligned} |\Lambda^+(\zeta)| &\ge \lambda^2 + \sum_{i=1}^{n(\zeta)} (\lambda(r_i(\zeta) + q_i(\zeta)) - r_i(\zeta) - q_i(\zeta)) - \frac{10-2h}{4h} + L(\zeta) \\ &\ge \lambda^2 - \frac{5}{4}\lambda + \sum_{i=1}^{n(\zeta)} (\lambda - 1)(r_i(\zeta) + q_i(\zeta)) \ge \lambda^2 - \frac{5}{4}\lambda \end{aligned} \tag{4.41}$$

where we also used $\lambda - 1 \ge 0$.

We next bound from above $|\bar{\Lambda}(\eta)|$. We first note that by using (4.40) and (4.12) we get

$$|\bar{\Lambda}(\eta)| \le |\Lambda^+(\overline{B}\,\overline{A}\eta)| + 5L(\eta). \tag{4.42}$$

Now, suppose that $\Lambda^+(\overline{B}\,\overline{A}\eta) \ne -\underline{1}$; by using (4.19) with $k = \lambda$ and exploiting $\eta \in \mathcal{A}_2$, we conclude

$$|\bar{\Lambda}(\eta)| \le \frac{1}{2}\lambda p(\eta) + 20\lambda + 210. \tag{4.43}$$

Suppose, on the other hand, that $\Lambda^+(\overline{B}\,\overline{A}\eta) = -\underline{1}$. By using (4.42), we get $|\bar{\Lambda}(\eta)| \le 5L(\eta) \le 20\lambda + 210$; hence the bound (4.43) holds since in this case $p(\eta) = 0$.

We finally bound $\Delta(\eta, \zeta)$ by using the preliminary inequalities (4.39), (4.41), and (4.43); we have

$$\Delta(\eta, \zeta) \ge 2(1-h)\left[\lambda^2 - \frac{85}{4}\lambda - \frac{1}{2}\lambda p(\eta) - 210\right]. \tag{4.44}$$

Recall $\eta \in \mathcal{A}_2 \setminus \mathcal{A}_3$, then $p(\eta) \le 2\lambda - 51$; hence by using (4.38), (4.44), and (2.13), we get

$$E(\eta, \zeta) - E(-\underline{1}) > \Gamma + \frac{1}{h} - \frac{53}{2} + O(h) > \Gamma$$

where in the last inequality we have chosen $h > 0$ small enough.

Step 6. Let $\eta \in \mathcal{A}_3 \setminus \mathcal{A}_4$ and $\zeta \in \mathcal{B}_2$. By using (4.33), Lemma 4.1, $N_A(\eta) \ge 0$, $N_B(\eta) \ge 0$, (2.25), and $\Delta(\eta, \zeta) \ge 0$, we get

$$E(\eta, \zeta) - E(-\underline{1}) \ge E(\overline{B}\,\overline{A}\eta) - E(-\underline{1}) = -4h|\Lambda^+(\overline{B}\,\overline{A}\eta)| + p(\eta). \tag{4.45}$$

Now, since $\eta \in \mathcal{A}_3 \setminus \mathcal{A}_4$, we can use (4.20) to obtain

$$E(\eta,\zeta) - E(-\underline{1}) \geq -h(p(\eta))^2 + (4h+8)p(\eta). \tag{4.46}$$

Finally, by exploiting the properties of the parabola on the right-hand side of (4.46) and recalling that for $\eta \in \mathcal{A}_3 \setminus \mathcal{A}_4$ the semi-perimeter satisfies the bounds $2\lambda - 50 \leq p(\eta) \leq 2\lambda + 4$, it is immediate to prove that the parabola attains its minimum at $p(\eta) = 2\lambda - 50$; hence, by using (4.46) and (2.12), we get $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$ for $h > 0$ small enough.

Step 7. Let $\eta \in \mathcal{A}_4 \setminus \mathcal{A}_5$ and $\zeta \in \mathcal{B}_2$. By using (4.33), (4.3), and $N_B(\eta) \geq 0$, we get the bound

$$E(\eta,\zeta) - E(-\underline{1}) \geq [E(\eta) - E(\overline{A}\eta)] + [E(\overline{B}\,\overline{A}\eta) - E(-\underline{1})] + \Delta(\eta,\zeta). \tag{4.47}$$

Since $\eta \in \mathcal{A}_4 \setminus \mathcal{A}_5$, we have that $n(\eta) = 1$ and then $2\lambda - 50 \leq p(\eta) \leq 2\lambda - 2$. We repeat, now, the same argument used at Step 6, but, since $n(\eta) = 1$, we have to use (4.18) instead of (4.20); we then get

$$E(\eta,\zeta) - E(-\underline{1}) \geq [E(\eta) - E(\overline{A}\eta)] + [\Gamma - 10 + O(h)] + \Delta(\eta,\zeta). \tag{4.48}$$

Moreover, since $n(\eta) = 1$ and $p(\eta) \leq 2\lambda - 2$, by using the same arguments developed in the proof of (4.26), we get

$$|\Lambda^+(\zeta) \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)| \geq 3. \tag{4.49}$$

To complete the proof of the Step 7, we distinguish four cases by means of the parameter $L(\eta)$. Consider, first, the case $L(\eta) \geq 3$; by using (4.48), (4.13), and $\Delta(\eta,\zeta) \geq 0$, it follows immediately $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$.

Consider, now, the case $L(\eta) = 2$. We first note that by using (4.48) and (4.13) we get

$$E(\eta,\zeta) - E(-\underline{1}) \geq 10 - 6h + [\Gamma - 10 + O(h)] + \Delta(\eta,\zeta) \geq \Gamma + \Delta(\eta,\zeta) + O(h). \tag{4.50}$$

The result $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$ will then be proven once we shall have obtained the bound $\Delta(\eta,\zeta) \geq 2(1-h)$.

To prove such a bound, we note that there exist $x, y \in \Lambda^+(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)$; since $x, y \in \Lambda \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)$, by the definition of the two maps A and B, it follows that they cannot be both stable w.r.t. η (see Sect. 3). If one of the two sites x and y, say x, is stable w.r.t. η, it is immediate to prove that $x \in \partial\Lambda^+(\overline{B}\,\overline{A}\eta)$ and $\{y\} = \partial\{x\} \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)$. Since x and y are nearest neighbors, it follows that there exist no site in $\Lambda^-(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)$ which is unstable w.r.t. η; hence, by using (4.49) and (2.20), it follows that $\Delta(\eta,\zeta) \geq 2(1-h)$. We consider, now, the case when both x and y are unstable w.r.t. η. Suppose that either $\zeta(x) = +1$ or $\zeta(y) = +1$; from (2.20) we have $\Delta(\eta,\zeta) \geq 2(1-h)$. On the other hand, if $\zeta(x) = \zeta(y) = -1$, it is easy to see that, since $L(\eta) = 2$, we have $|\Lambda_{\mathrm{u}}^-(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)| \leq 1$ (recall the definition (3.3)). Then, by using (4.49), it follows that $\Delta \geq 2(1-h)$.

Consider, now, the case $L(\eta) = 1$. We first note that, by using (4.48) and (4.13), we get

$$E(\eta,\zeta) - E(-\underline{1}) \geq 6 - 2h + [\Gamma - 10 + O(h)] + \Delta(\eta,\zeta) \geq \Gamma - 4 + \Delta(\eta,\zeta) + O(h). \tag{4.51}$$

The result $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$ will then be proven once we shall have obtained the bound $\Delta(\eta,\zeta) \geq 3 \cdot 2(1-h)$.

To prove such a bound we let x the site such that $\{x\} := \Lambda^+(\eta) \setminus \Lambda^+(\overline{B}\,\overline{A}\eta)$. Suppose $\zeta(x) = +1$; since x is unstable w.r.t. η, by (2.20) and (4.49), we have $\Delta(\eta,\zeta) \geq 2(1-h) +$

$2(1-h)+2(1-h)$. On the other hand, suppose $\zeta(x)=-1$; since $|\Lambda^-_{\mathrm{u}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|=0$, by (4.49) it follows that $\Delta(\eta,\zeta)\geq 2(1-h)+2(1-h)+2(3-h)$.

Consider, finally, the case $L(\eta)=0$. Recall (2.20); the condition (4.49) implies that $\Delta(\eta,\zeta)\geq 3\cdot 2(3-h)$. Hence, by using also (4.48), (4.2), and $N_A(\eta)\geq 0$, we get

$$E(\eta,\zeta)-E(-\underline{1})\geq \Gamma-10+O(h)+3\cdot 2(3-h)>\Gamma.$$

Step 8. Let $\eta\in\mathcal{A}_5$ and $\zeta\in\mathcal{B}_2$. We remark that, since $p(\eta)=2\lambda-1$ and $\ell_1\vee m_1\leq\lambda$, we have $\overline{B}\,\overline{A}\eta\in\mathcal{R}$. Hence, by using (4.33), (4.3), and (3.24), we get

$$E(\eta,\zeta)-E(-\underline{1})\geq[E(\eta)-E(\overline{A}\eta)]+4hN_B(\eta)+[\Gamma-10+6h]+\Delta(\eta,\zeta). \tag{4.52}$$

To complete the proof of Step 8, we distinguish four cases by means of the parameter $L(\eta)$. Consider, first, the case $L(\eta)\geq 3$; by using (4.52), (4.13), $N_B(\eta)\geq 0$, and $\Delta(\eta,\zeta)\geq 0$, we get

$$E(\eta,\zeta)-E(-\underline{1})\geq 12-16h+\Gamma-10+6h>\Gamma.$$

Consider, now, the case $L(\eta)=2$; we let x,y be the two sites in $\Lambda^+(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$. We first note that, by using the inequalities (4.52) and (4.13), we get the bound

$$E(\eta,\zeta)-E(-\underline{1})\geq 4hN_B(\eta)+\Gamma+\Delta(\eta,\zeta). \tag{4.53}$$

Suppose, first, $N_B(\eta)\geq 1$; by (4.53) and $\Delta(\eta,\zeta)\geq 0$, we immediately get $E(\eta,\zeta)-E(-\underline{1})>\Gamma$. We are then left with the case $N_B(\eta)=0$, i.e., $\Lambda^+(\eta)\supset\Lambda^+(\overline{B}\,\overline{A}\eta)$; by using (4.53), the result $E(\eta,\zeta)-E(-\underline{1})>\Gamma$ will be proven once we shall have obtained the bound $\Delta(\eta,\zeta)\geq 2(1-h)$.

We note that $|\Lambda^+_{\mathrm{s}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|\leq 1$, indeed if by the way of contradiction x and y belonged both to $\Lambda^+_{\mathrm{s}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$, then it should necessarily be $x,y\in\partial\Lambda^+(\overline{B}\,\overline{A}\eta)$ and $\mathrm{d}(x,y)=1$, namely, there would be a two-site protuberance added to the $\lambda\times(\lambda-1)$ rectangle of pluses which is present in η. Hence, we would have $\eta\in\mathcal{C}\subset\mathcal{G}^{\mathrm{c}}$, which is a contradiction.

Suppose $|\Lambda^+_{\mathrm{s}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|=1$ and let x be the site in $\Lambda^+_{\mathrm{s}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$; since x is stable w.r.t. η, we must necessarily have $x\in\partial\Lambda^+(\overline{B}\,\overline{A}\eta)$ and $y\in\partial\{x\}\setminus\overline{\Lambda^+(\overline{B}\,\overline{A}\eta)}$. Note, also, that this implies $|\Lambda^-_{\mathrm{u}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|=0$. Thus, for $\zeta(y)=+1$, in the sum in (2.20) there is at least the term corresponding to y; then we have $\Delta(\eta,\zeta)\geq 2(1-h)$. On the other hand, if it were $\zeta(y)=-1$, recalling (4.26) we would have that in the sum in (2.20) there is at least a term corresponding to the flip of the spin associated with a site in $\Lambda^-(\eta)$ which is stable w.r.t. η, hence we would have $\Delta(\eta,\zeta)\geq 2(1-h)$. Suppose, finally, $|\Lambda^+_{\mathrm{s}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|=0$; it is immediate to prove that $|\Lambda^-_{\mathrm{u}}(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|\leq 1$. Then we get $\Delta(\eta,\zeta)\geq 2(1-h)$, since from (4.26) it follows that in the sum in (2.20) there is at least a term corresponding to the persistence of the spin associated with a site in $\Lambda^+_{\mathrm{u}}(\eta)$ or to the flip of the spin associated with a site in $\Lambda^-(\eta)$ which is stable w.r.t. η.

Consider, now, the case $L(\eta)=1$. We let x be the site in $\Lambda^+(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$, note that x in unstable w.r.t. η and w is stable w.r.t. η for any $w\in\Lambda^-(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$. We remark that by using (4.52), (4.13), (4.3), and $N_B(\eta)\geq 0$, we get the bound

$$E(\eta,\zeta)-E(-\underline{1})\geq\Gamma-4+4h+\Delta(\eta,\zeta) \tag{4.54}$$

and distinguish different cases depending on the number of plus spins in the configuration ζ which are associated to sites outside the support of the configuration $\overline{B}\,\overline{A}\eta$, that is on $|\Lambda^+(\zeta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|\geq 2$, see (4.26).

Suppose, first, $|\Lambda^+(\zeta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|\ge 3$; since $x\in\Lambda^+_{\mathrm{u}}(\eta)$ and w is stable w.r.t. η for any $w\in\Lambda^-(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$, we have $\Delta(\eta,\zeta)\ge 3\cdot 2(1-h)$, for in the sum in (2.20) there are at least three terms.

We are left with the case $|\Lambda^+(\zeta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|=2$; we let $\{y,z\}:=\Lambda^+(\zeta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$ and notice that it must be necessarily $y,z\in\partial\Lambda^+(\overline{B}\,\overline{A}\eta)$ and $\mathrm{d}(y,z)=1$, otherwise it would be $\zeta\in\mathcal{G}$. Suppose, first, $\zeta(x)=-1$; since $x\ne y$, $x\ne z$, and y and z are nearest neighbors, it follows that at most one of the two sites y and z is nearest neighbor of x. Then, since in the sum (2.20) there are at least two terms and one of them is greater or equal to $2(3-h)$, we have $\Delta(\eta,\zeta)\ge 2(1-h)+2(3-h)$. By the previous inequality and (4.54) we get $E(\eta,\zeta)-E(-\underline{1})>\Gamma$. Suppose, finally, $\zeta(x)=+1$; without loss of generality we let $y=x$. Since $z\in\partial\{x\}\cap\partial\Lambda^+(\overline{B}\,\overline{A}\eta)$, by (2.20) we have $\Delta(\eta,\zeta)\ge 2(1-h)+2(1-h)$, with one of the two terms corresponding to the persistence of the plus spin associated to x in η and the other corresponding to the flip of the minus spin associated to z in η. By the previous inequality and (4.54) we get $E(\eta,\zeta)-E(-\underline{1})\ge\Gamma$.

Consider, finally, the case $L(\eta)=0$. By using (4.52), (4.2), $N_A(\eta)\ge 0$, $N_B(\eta)\ge 0$, and (4.27), we get

$$E(\eta,\zeta)-E(-\underline{1})\ge\Gamma-10+6h+12-4h>\Gamma.$$

Item 3. Suppose $\zeta\in\mathcal{C}$ and $\eta\in\pi(\zeta)$, by using (2.9) and (2.20) it follows $E(\eta,\zeta)-E(-\underline{1})=\Gamma$.

Conversely, suppose $\eta\in\mathcal{G}$ and $\zeta\in\mathcal{G}^{\mathrm{c}}$ such that $E(\eta,\zeta)-E(-\underline{1})=\Gamma$. By using the results in the proof of Item 2 above, see in particular Step 8, we have that it must be necessarily $\eta\in\mathcal{A}_5$, $\zeta\in\mathcal{B}_2$, $L(\eta)=1$, $|\Lambda^+(\zeta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)|=2$, and $\zeta(x)=+1$, with x such that $\{x\}=\Lambda^+(\eta)\setminus\Lambda^+(\overline{B}\,\overline{A}\eta)$, indeed for any different choice of η and ζ it has be proven $E(\eta,\zeta)-E(-\underline{1})>\Gamma$. For configurations η and ζ as above we have also proven that $\overline{B}\,\overline{A}\eta\in\mathcal{R}$, that $\Delta(\eta,\zeta)\ge 2\cdot 2(1-h)$, and that there exists $z\in\partial\{x\}\cap\partial\Lambda^+(\overline{B}\,\overline{A}\eta)$ such that $\zeta(z)=+1$.

Now, by using $\overline{B}\,\overline{A}\eta\in\mathcal{R}$, $\Delta(\eta,\zeta)\ge 2\cdot 2(1-h)$, (4.33), (4.13), (4.3), and (3.24), we get

$$E(\eta,\zeta)-E(-\underline{1})\ge 6-2h+4hN_B(\eta)+\Gamma-10+6h+2\cdot 2(1-h)=\Gamma+4hN_B(\eta).$$

If it were $N_B(\eta)\ge 1$ it would follow $E(\eta,\zeta)-E(-\underline{1})>\Gamma$, then it must necessarily be $N_B(\eta)=0$.

By the above characterization of η we have that $\eta\in\mathcal{P}$; then, by using (4.33) and the definition of the map A we get the following expression for the communication energy $E(\eta,\zeta)$:

$$E(\eta,\zeta)-E(-\underline{1})=6-2h+\Gamma-10+6h+\Delta(\eta,\zeta)=\Gamma-4+4h+\Delta(\eta,\zeta).$$

Since $\zeta(x)=\zeta(z)=+1$, we have that $\Delta(\eta,\zeta)=2\cdot 2(1-h)$ if and only if $\zeta(w)=+1$ for all $w\in\Lambda^+(\overline{B}\,\overline{A}\eta)$. We then have that $\zeta\in\mathcal{C}$ and $\eta\in\pi(\zeta)$. □

Appendix: Review of Results in [9]

The proof of Theorem 2.1 is based on general results in [9, Theorem 4.1, 4.9, and 5.4] concerning the hitting time on the set of global minima of the energy for the chain started at a metastable state. We restate those results in our framework which is slightly different from the one considered in that paper (see the discussion at the beginning of Sect. 2.8).

Recall (2.27). Let $\mathcal{S}^{\mathrm{s}}$ be the set of global minima of the energy (2.9). For any $\sigma \in \mathcal{S}$, let $\mathcal{I}_\sigma := \{\eta \in \mathcal{S} : E(\eta) < E(\sigma)\}$ be the set of states with energy below $E(\sigma)$ and $V_\sigma := \Phi(\sigma, \mathcal{I}_\sigma) - E(\sigma)$ be the *stability level of* σ. Set $V_\sigma := \infty$ if $\mathcal{I}_\sigma = \emptyset$. We define the set of *metastable states* $\mathcal{S}^{\mathrm{m}} := \{\eta \in \mathcal{S} : V_\eta = \max_{\sigma \in \mathcal{S}\setminus\mathcal{S}^{\mathrm{s}}} V_\sigma\}$. We say that $\mathcal{W}(\eta, \zeta) \subset \mathcal{S}$ is a *gate* for the transition from $\eta \in \mathcal{S}$ to $\zeta \in \mathcal{S}$ if and only if the two following conditions are satisfied: (1) for any $\sigma \in \mathcal{W}(\eta, \zeta)$ there exist a path $\omega \in \Theta(\eta, \zeta)$, such that $\Phi_\omega = \Phi(\eta, \zeta)$, and $i \in \{2, \ldots, |\omega|\}$ such that $\omega_i = \sigma$ and $E(\omega_{i-1}, \omega_i) = \Phi(\eta, \zeta)$; (2) $\omega \cap \mathcal{W}(\eta, \zeta) \neq \emptyset$ for any path $\omega = \Theta(\eta, \zeta)$ such that $\Phi_\omega = \Phi(\eta, \zeta)$. A function $f : \beta \in \mathbb{R} \to f(\beta) \in \mathbb{R}$ is called *super-exponentially small* (SES) in the limit $\beta \to \infty$ if and only if $\lim_{\beta\to\infty}(1/\beta)\log f(\beta) = -\infty$. Given $\sigma \in \mathcal{S}$ and $A \subset \mathcal{S}$, finally, recall the definition of hitting time τ_A^σ given in (2.4) and the notation $\mathbb{E}_\sigma$ introduced just before it.

Theorem A.1 (restatement of Theorem 4.1 in [9]) *Let* $\sigma \in \mathcal{S}^{\mathrm{m}}$; *for any* $\delta > 0$, *there exist* $\beta_0 > 0$ *and* $K > 0$ *such that, for any* $\beta > \beta_0$,

$$\mathbb{P}_\sigma(\tau_{\mathcal{S}^{\mathrm{s}}}^\sigma < e^{\beta V_\sigma - \beta\delta}) < e^{-K\beta} \tag{A.1}$$

and

$$\mathbb{P}_\sigma(\tau_{\mathcal{S}^{\mathrm{s}}}^\sigma > e^{\beta V_\sigma + \beta\delta}) = \mathrm{SES}. \tag{A.2}$$

Theorem A.2 (restatement of Theorem 4.9 in [9]) *Let* $\sigma \in \mathcal{S}^{\mathrm{m}}$, *then*

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}_\sigma[\tau_{\mathcal{S}^{\mathrm{s}}}^\sigma] = V_\sigma. \tag{A.3}$$

Theorem A.3 (restatement of Theorem 5.4 in [9]) *Let* $\sigma, \eta \in \mathcal{S}$; *consider a gate* $\mathcal{W}$ *for the transition from* σ *to* η. *Then there exists* $c > 0$ *such that*

$$\mathbb{P}_\sigma(\tau_{\mathcal{W}}^\sigma > \tau_\eta^\sigma) \le e^{-\beta c} \tag{A.4}$$

for β *large enough.*

The proof of Theorems A.1–A.3 can be achieved by arguments much similar to the one developed in [9]. To this purpose the main ingredients are the revised definition of cycle and the revised statement of [9, Theorem 2.17] (see also [12, Theorem 6.23]) and [9, Theorem 3.1].

Let $A \subset \mathcal{S}$, consider $\Phi(A, A^{\mathrm{c}})$, we say that A is a *cycle* if and only if

$$\max_{\sigma,\eta\in A} \Phi(\sigma, \eta) < \Phi(A, A^{\mathrm{c}}). \tag{A.5}$$

Let $\sigma \in \mathcal{S}$; we say that the *singleton* $\{\sigma\} \subset \mathcal{S}$ is a *trivial cycle* if and only if it is not a cycle. Given a cycle $A \subset \mathcal{S}$, we denote by $F(A)$ the set of the minima of the energy in A, i.e.,

$$F(A) := \{\sigma \in A : \min_{\eta\in A} E(\eta) = E(\sigma)\}. \tag{A.6}$$

We also write $E(F(A)) = E(\sigma)$ with $\sigma \in F(A)$. Noted that $\Phi(\sigma, A^{\mathrm{c}}) = \Phi(\sigma', A^{\mathrm{c}})$ for any $\sigma, \sigma' \in F(A)$, we pick $\sigma \in F(A)$ and set $\Phi(A) := \Phi(\sigma, A^{\mathrm{c}})$.

Theorem A.4 (restatement of Theorem 2.17 [9]) *Let $A \subset \mathcal{S}$ be a cycle. For any $\sigma \in A$, $\eta \in A^{\mathrm{c}}$, $\epsilon, \epsilon' > 0$, $\delta \in (0, \epsilon)$, and $\beta > 0$ large enough*

$$\mathbb{P}_\sigma(\tau^\sigma_{A^{\mathrm{c}}} < e^{\beta[\Phi(A)-E(F(A))]+\beta\epsilon};\ \tau^\sigma_{A^{\mathrm{c}}} = \tau^\sigma_\eta) \geq e^{-\beta[\Phi(\eta,A))-\Phi(A)]-\beta\epsilon'} \tag{A.7}$$

and

$$\mathbb{P}_\sigma(\tau^\sigma_{A^{\mathrm{c}}} > e^{\beta[\Phi(A)-E(F(A))]-\beta\epsilon}) \geq 1 - e^{-\beta\delta}. \tag{A.8}$$

Moreover, there exists $\kappa > 0$ such that for any $\sigma, \sigma' \in A$ and β large enough

$$\mathbb{P}_\sigma(\tau^\sigma_{\sigma'} < \tau^\sigma_{A^{\mathrm{c}}}) \geq 1 - e^{-\beta\kappa}. \tag{A.9}$$

Equation (A.7) is a bound from below to the probability that the chain exits a cycle A in a time smaller than the exponential of β times the height $\Phi(A) - E(F(A))$ of the cycle plus ϵ. In particular, it is stated that for such a probability the estimate is optimal when the exit from the cycle is achieved by touching a configuration η such that $\Phi(\eta, A)$ is equal to $\Phi(A)$. In this case, for any ϵ', provided β is chosen large enough, such a probability is larger than $\exp\{-\beta\epsilon'\}$. Equation (A.8) is a bound from below to the probability that the chain exits a cycle A in a time larger than the exponential of β times the height $\Phi(A) - E(F(A))$ of the cycle minus ϵ; such a probability is larger than $1 - \exp\{-\beta\delta\}$ with $\delta \in (0, \epsilon)$. Equation (A.9) is a bound from below to the probability that the chain started at $\sigma \in A$ visits another configuration σ' belonging to the cycle A before exiting it; such a probability is larger than $1 - \exp\{-\beta\kappa\}$ with $\kappa > 0$ not depending on σ and σ'.

Before stating the revised version of [9, Theorem 3.1], we introduce the concept of metastable state at level $V \in \mathbb{R}$. We call *metastable set at level* $V \in \mathbb{R}$ the set of all states with stability level strictly larger than V, i.e., $\mathcal{S}_V := \{\sigma \in \mathcal{S} : V_\sigma > V\}$. Any $\sigma \in \mathcal{S}_V$ is such that for any path ω starting from σ and ending in a configuration with energy lower than $E(\sigma)$, the quantity $\Phi_\omega - E(\sigma)$, that is the energy level reached along the path and measured with respect to σ, is lower than V.

Theorem A.5 (restatement of Theorem 3.1 in [9]) *For any $\epsilon > 0$ and $\beta > 0$ large enough*

$$\sup_{\sigma \in \mathcal{S}} \mathbb{P}_\sigma(\tau^\sigma_{\mathcal{S}_V} > e^{\beta V + \beta\epsilon}) = \mathrm{SES}. \tag{A.10}$$

Equation (A.10) states that the probability that the chain started at $\sigma \in \mathcal{S}$ visits a configuration with metastability level $V \in \mathbb{R}$ in a time larger than the exponential of β times V plus ϵ is super-exponentially small in β.

The proof of Theorem A.5 can be achieved by repeating the same arguments developed in [9] and based on [9, Theorem 2.17]. The proof of Theorem A.4 can be achieved by repeating the same arguments quoted in [9] and developed in the proof of [12, Theorem 6.23] (see also [11, Proposition 3.7]). In particular, in the proof of (A.8), a revised version of the so called *reversibility property* [11, Lemma 3.1] is needed.

Lemma A.6 *Let $A \subset \mathcal{S}$ be a cycle. For any $\sigma \in F(A)$ and $\epsilon > 0$, there exist $\beta_0 > 0$ and $c > 0$, such that for any $\beta \geq \beta_0$,*

$$\mathbb{P}_\sigma\left(\tau^\sigma_{A^{\mathrm{c}}} \leq e^{\beta[\Phi(A)-E(F(A))]-\beta\epsilon}\right) \leq e^{-c\beta}. \tag{A.11}$$

To prove Lemma A.6 we develop an argument much similar to the one proposed by Olivieri and Vares to prove [12, Lemma 6.22]. The difference between the two cases is in the fact that the Hamiltonian of the model (2.1) of the present paper depends on the inverse temperature β, while [12, Lemma 6.22] refers to a model (see [12, Condition R in Chapter 6]) with Hamiltonian not depending on the temperature. Note, also, that the reversibility statement (A.11) is given in terms of energy-like quantities not depending on β; on the other hand in the proof of the lemma the key property is the reversibility of the dynamics w.r.t. the Hamiltonian (2.5).

Proof of Lemma A.6. First of all we make explicit how hamiltonian-like quantities differ from the corresponding energy-like quantities multiplied times β. More precisely, given $\sigma, \eta \in \mathcal{S}$, we have

$$H(\sigma) = \beta\, E(\sigma) - \sum_{x\in\Lambda} \log\left[1 + e^{-2\beta|S_\sigma(x)+h|}\right] + |\Lambda| \log 2 \tag{A.12}$$

and

$$H(\sigma,\eta) = \beta E(\sigma,\eta) + |\Lambda| \log 2. \tag{A.13}$$

The equality (A.13) follows easily from (A.12) and (2.23). We prove, now, (A.12). Using (2.5), (2.9), and recalling that $\cosh x = \cosh(-x)$ for any $x \in \mathbb{R}$, we have that

$$\begin{aligned}
H(\sigma) - \beta(\sigma) &= -\sum_{x\in\Lambda} \log\cosh(\beta|S_\sigma(x)+h|) + \beta \sum_{x\in\Lambda} |S_\sigma(x)+h| \\
&= -\sum_{x\in\Lambda} [\log\cosh(\beta|S_\sigma(x)+h|) - \log\exp(\beta|S_\sigma(x)+h|)] \\
&= -\sum_{x\in\Lambda} \log \frac{e^{\beta|S_\sigma(x)+h|} + e^{-\beta|S_\sigma(x)+h|}}{2e^{\beta|S_\sigma(x)+h|}} = -\sum_{x\in\Lambda} \log \frac{1+e^{-2\beta|S_\sigma(x)+h|}}{2}
\end{aligned}$$

which yields (A.12).

Consider, now, an integer $T \geq 2$; recalling that the chain is reversible with respect to the Gibbs measure μ defined above (2.5), we have that

$$\begin{aligned}
\mathbb{P}_\sigma(\tau_{A^c} \leq T) &= \sum_{\xi\in A^c} \Bigg[p(\sigma,\xi) + \sum_{n=1}^{T-1} \sum_{\xi_1,\dots,\xi_n\in A} p(\sigma,\xi_1)\cdots p(\xi_{n-1},\xi_n) p(\xi_n,\xi) \Bigg] \\
&= \sum_{\xi\in A^c} \Bigg[p(\sigma,\xi) + \sum_{n=1}^{T-1} \sum_{\xi_1,\dots,\xi_n\in A} e^{-[H(\xi_n)-H(\sigma)} p(\xi_n,\xi_{n-1})\cdots p(\xi_1,\sigma) p(\xi_n,\xi) \Bigg]
\end{aligned} \tag{A.14}$$

where the detailed balance (2.6) has been used to invert the order of the configurations in the terms $p(\sigma,\xi_1),\dots,p(\xi_{n-1},\xi_n)$. By using the definition (2.17) of transition Hamiltonian, we have that $\exp\{-[H(\xi_n) - H(\sigma)]\}\, p(\xi_n,\xi) = \exp\{-[H(\xi_n,\xi) - H(\sigma)]\}$. Noting, also, that

$$\sum_{\xi_1,\dots,\xi_{n-1}\in A} p(\xi_n,\xi_{n-1})\cdots p(\xi_1,\sigma) = \mathbb{P}_\sigma(\sigma_n = \xi_n)$$

we have

$$\begin{aligned}
\mathbb{P}_\sigma(\tau_{A^c} \le T) &= \sum_{\xi\in A^c}\Big[e^{-[H(\sigma,\xi)-H(\sigma)} + \sum_{n=1}^{T-1}\sum_{\xi_n\in A} e^{-[H(\xi_n,\xi)-H(\sigma)}\mathbb{P}_\sigma(\sigma_n=\xi_n)\Big] \\
&\le e^{H(\sigma)}\sum_{\xi\in A^c}\big[e^{-H(\sigma,\xi)} + \sum_{n=1}^{T-1}\sum_{\xi_n\in A} e^{-H(\xi_n,\xi)}\big] \\
&= e^{H(\sigma)}\sum_{\xi\in A^c}\Big[e^{-H(\sigma,\xi)} + (T-1)\sum_{\zeta\in A} e^{-H(\zeta,\xi)}\Big] \le e^{H(\sigma)}T\sum_{\zeta\in A,\xi\in A^c} e^{-H(\zeta,\xi)}
\end{aligned}$$

which yields

$$\mathbb{P}_\sigma(\tau_{A^c} \le T) \le e^{H(\sigma)}\,T\,|A|\,|A^c|\max_{\zeta\in A,\xi\in A^c} e^{-H(\zeta,\xi)} = e^{H(\sigma)}T|A||A^c|e^{-\min_{\zeta\in A,\xi\in A^c} H(\zeta,\xi)}. \tag{A.15}$$

By using, finally, (A.12), (A.13), and (A.15), we get

$$\mathbb{P}_\sigma(\tau_{A^c} \le T) \le e^{\beta E(\sigma)}T|A||A^c|e^{-\min_{\zeta\in A,\xi\in A^c} E(\zeta,\xi)}\exp\Big\{-\sum_{x\in\Lambda}\log\big[1+e^{-2\beta|S_\sigma(x)+h|}\big]\Big\}. \tag{A.16}$$

Since A is a cycle, we have that $\min_{\zeta\in A,\xi\in A^c} E(\zeta,\xi) = \Phi(A)$. Hence, remarked that

$$0 < \exp\Big\{-\sum_{x\in\Lambda}\log\big[1+e^{-2\beta|S_\sigma(x)+h|}\big]\Big\} \le (1+\exp\{-2\beta(5+h)\})^{-|\Lambda|} \le \left(\frac{3}{2}\right)^{|\Lambda|}$$

for β large enough, the bound (A.8) follows once we take $T = e^{\beta[\Phi(A)-E(F(A))]-\beta\epsilon}$. □

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